Exercise 1

A single-link robot with a rotary joint is motionless at $\theta = 0$. It is desired to move the joint in a smooth manner to $\theta = \frac{\pi}{2}$ in one second. Find the cubic polynomial coefficients that accomplish the desired motion and brings the manipulator to rest at the desired destination.

Solution

Let us write the cubic as,

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

And the derivative with respect to $t$ as,

$$\dot{\theta}(t) = a_1 + 2 a_2 t + 3 a_3 t^2$$

Next let us use the initial conditions,

$$\theta(t = 0) = a_0 = 0$$
$$\dot{\theta}(t = 0) = a_1 = 0$$

Using the conditions on the target and the previous results we obtain the following two simple equations,

$$\theta(t = 1) = a_2 + a_3 = \frac{\pi}{2}$$
$$\dot{\theta}(t = 1) = 2 a_2 + 3 a_3 = 0$$

By solving the last system of equations we obtain $a_3 = -\pi, a_2 = \frac{3}{2}\pi$ so that the cubic polynomial is

$$\theta(t) = \frac{3}{2}\pi t^2 - \pi t^3$$
$$\dot{\theta}(t) = 3\pi t - 3\pi t^2$$
**Exercise 2**

A single-link robot with a rotary joint is motionless at $\theta = 0$. It is desired to move the joint in a smooth manner to $\theta = \frac{\pi}{2}$ at $t = 1$ and to $\theta = \frac{3\pi}{2}$ at $t = 2$. Find the cubic polynomial coefficients that accomplish the desired motion and brings the manipulator to rest at the desired destination at $t = 2$. Note that there is no constraint on the velocity at the intermediate point, however, the motion must be smooth.

**Solution**

We wish to pass through the intermediate point at $t = 1$ smoothly and without stopping. Therefore, we should constraint the joint angle and its first and second derivative to be equal at the intermediate point.

Let us write the cubic as,

\[
\theta_1(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\
\theta_2(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3
\]

And the derivatives with respect to $t$ as,

\[
\dot{\theta}_1(t) = a_1 + 2a_2 t + 3a_3 t^2 \\
\dot{\theta}_2(t) = b_1 + 2b_2 t + 3b_3 t^2 \\
\ddot{\theta}_1(t) = 2a_2 + 6a_3 t \\
\ddot{\theta}_2(t) = 2b_2 + 6b_3 t
\]

Using the initial conditions,

\[
\theta_1(t = 0) = a_0 = 0 \\
\dot{\theta}_1(t = 0) = a_1 = 0
\]

Constraint on the joint angle and its derivatives at $t = 1$ give us,

\[
\theta_1(t = 1) = a_2 + a_3 = \frac{\pi}{2} \\
\theta_2(t = 1) = b_0 + b_1 + b_2 + b_3 = \frac{3\pi}{2} \\
\dot{\theta}_1(t = 1) = \dot{\theta}_2(t = 1) \implies 2a_2 t + 3a_3 - b_1 - 2b_2 t - 3b_3 t^2 = 0 \\
\ddot{\theta}_1(t = 1) = \ddot{\theta}_2(t = 1) \implies 2a_2 + 6a_3 - 2b_2 - 6b_3 = 0
\]

Constraint on the joint angle and its derivatives at $t = 2$ give us

\[
\theta_2(t = 2) = b_0 + 2b_1 + 4b_2 + 8b_3 = \frac{3\pi}{2} \\
\dot{\theta}_2(t = 2) = b_1 + 4b_2 + 12b_3 = 0
\]
We can write the system of linear equations in vectorized form to calculate our six variables,

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 \\
2 & 3 & 0 & -1 & -2 & -3 \\
2 & 6 & 0 & 0 & -2 & -6 \\
0 & 0 & 1 & 2 & 4 & 8 \\
0 & 0 & 0 & 1 & 4 & 12 \\
\end{bmatrix}
\begin{bmatrix}
a_2 \\
a_3 \\
b_0 \\
b_1 \\
b_2 \\
b_3 \\
\end{bmatrix}
\begin{bmatrix}
\pi/2 \\
\pi \\
0 \\
0 \\
3\pi/2 \\
0 \\
\end{bmatrix}
\]

So that,

\[
\begin{bmatrix}
a_2 \\
a_3 \\
b_0 \\
b_1 \\
b_2 \\
b_3 \\
\end{bmatrix}
= \begin{bmatrix}
1.1781 \\
0.3927 \\
3.1416 \\
-9.4248 \\
10.6029 \\
-2.7489 \\
\end{bmatrix}
\]

Let us plot the trajectory to verify that it is continuous and smooth,
A = [1 1 0 0 0 0; 
0 0 1 1 1 1; 
2 3 0 -1 -2 -3; 
2 6 0 0 -2 -6; 
0 0 1 2 4 8; 
0 0 0 1 4 12 ];
b = [0.5*pi; 0.5*pi; 0; 0; 1.5*pi; 0];
x = A\b;
t0 = 0:0.01:1;
Theta1 = x(1)*t0.^2 + x(2)*t0.^3;
Theta1_Dot = 2*x(1)*t0 + 3*x(2)*t0.^2;
Theta1_Dot2 = 2*x(1) + 6*x(2)*t0;
t1 = 1:0.01:2;
Theta2 = x(3)+x(4)*t1 + x(5)*t1.^2 + x(6)*t1.^3;
Theta2_Dot = x(4) + 2*x(5)*t1 + 3*x(6)*t1.^2;
Theta2_Dot2 = 2*x(5) + 6*x(6)*t1;
figure(1);
suplot(3, 1, 1)
plot(t0, Theta1, 'b', t1, Theta2,'r');
ylabel('$\theta(t)[rad]$','interpreter','latex')
grid on
subplot(3, 1, 2)
plot(t0, Theta1_Dot, 'b', t1, Theta2_Dot,'r');
ylabel('$\dot{\theta(t)}[rad/sec]$','interpreter','latex')
grid on
subplot(3, 1, 3)
plot(t0, Theta1_Dot2, 'b', t1, Theta2_Dot2,'r');
ylabel('$\ddot{\theta(t)}[rad/{sec^2}]$','interpreter','latex')
xlabel('$t[sec]$','interpreter','latex')
grid on
Exercise 3 - Matlab

Given the following two-link robot with rotary joints, we desire to move the end effector linearly from $[2 \ 0 \ 0]^T$ to $[0 \ 2 \ 0]^T$ at constant speed by $t = 1[sec]$. Find $\theta_1(t), \theta_2(t)$ that solve satisfy the conditions above.

Solution

By calculating the forward kinematics we can find the position of the end effector with respect to $\theta_1, \theta_2$ from the right most column in the homogeneous transformation matrix $^3T_0$ so that,

$$
\begin{align*}
P &= \begin{bmatrix} 
\cos \theta_1 + \cos(\theta_1 + \theta_2) \\
\sin \theta_1 + \sin(\theta_1 + \theta_2) \\
0
\end{bmatrix} = \begin{bmatrix} 
\cos \theta_1 + \cos(\theta_1 + \theta_2) \\
\sin \theta_1 + \sin(\theta_1 + \theta_2) \\
0
\end{bmatrix} \\
\end{align*}
$$

Moreover the end effector is required to move linearly with constant speed, therefore, we can write the following linear interpolation equation with respect to time,

$$
P(t) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} (1 - t) + \begin{bmatrix} 0 \\ 2t \\ 0 \end{bmatrix}
$$

Next, we will compare the equation to obtain the following system equations,

$$
\begin{bmatrix} 
\cos \theta_1 + \cos(\theta_1 + \theta_2) \\
\sin \theta_1 + \sin(\theta_1 + \theta_2) \\
0
\end{bmatrix} = \begin{bmatrix} 
2(1 - t) \\
2t \\
0
\end{bmatrix}
$$
We can solve the system of equations easily using the Matlab Symbolic tool box and obtain an expression for the angles with respect to time,

\[ \theta_1(t) = 2 \arctan \left( \frac{t + \sqrt{2t(-2t^3 + 4t^2 - 3t + 1)}}{2t^2 - 3t + 2} \right) \]

\[ \theta_2(t) = -2 \arctan \left( \frac{2t(1-t)}{2t^2 - 2t + 1} \right) \]

The following program calculates the solution and plots the tractor below,

```matlab
%%
syms Theta1 Theta2 t real ;
assume( t >=0);
assume( t <=1);
S = solve ( [ cos(Theta1) + cos(Theta1+Theta2) == 2*(1-t),
              sin(Theta1) + sin(Theta1+Theta2)== 2*t ] , [ Theta1 , Theta2 ] ) ;
pretty(simplify(S.Theta1(1)))
pretty(simplify(S.Theta2(1)))
T = 0:0.01:1;
Pt = repmat((1-T),2,2).*[2;0] + repmat(T,2,2).*[0;2];
theta1(t) = S.Theta1(1);
theta2(t) = S.Theta2(1);
figure(2);
subplot(2,1,1)
plot(Pt(1,:),Pt(2,:),'b')
xlabel('${P_X(t)}$','interpreter','latex')
ylabel('${P_Y(t)}$','interpreter','latex')
subplot(2,1,2)
plot(T, theta1(T),'b',T, theta2(T),'r')
ylabel('${\theta(t)[rad]}$','interpreter','latex')
xlabel('${t[sec]}$','interpreter','latex')
legend('${\theta_1(t)[rad]}$','${\theta_2(t)[rad]}$','interpreter','latex')
grid on
```