IMA 2012 Flight Competition - Mission Description and Rules

Contents

1. General Information
2. Frequencies
3. Sizes and weight limitations
4. Scores (indoor and outdoor)
5. Indoor competition
   - Indoor dynamics competition
   - Indoor autonomy competition
Flying time and number of flights
The allowed flying time is 3 minutes. Each team will be given the chance to fly twice. The better score will be counted. The second flight time will be in the same sequence as in the first round.

Figure 5.3: Indoor mission
Inverse Kinematics

How do I put my hand here?

IK: Choose these angles!
Cartesian Space & Joint Space

Joint Space \((q_1, q_2, \ldots, q_N)\)

Cartesian Space \((x, y, z, \phi, \theta, \psi)\)

Forward kinematics

Inverse kinematics (arm solution) \(e^T - 1\)
Inverse Relationship?

• Find joint positions for a given end-effector pose

\[
q_1 = g_1(\phi_e, \psi_e, \theta_e, x_e, y_e, z_e) \\
q_2 = g_2(\phi_e, \psi_e, \theta_e, x_e, y_e, z_e) \\
\vdots \\
q_N = g_N(\phi_e, \psi_e, \theta_e, x_e, y_e, z_e)
\]

• Also find joint positions, speeds and accelerations for a sequence of poses
Solvability

- The kinematic equations are nonlinear

- Issues:
  - Existence of solutions
  - Multiple solutions
  - Method
The Workspace

**Workspace**

- **Workspace**: volume of space which can be reached by the end effector
  - **Dextrous workspace**: volume of space where the end effector can be arbitrarily oriented
  - **Reachable workspace**: volume of space which the robot can reach in at least one orientation
<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$L_1$</td>
<td>0</td>
<td>$\theta_2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$L_2$</td>
<td>0</td>
<td>$\theta_3$</td>
</tr>
</tbody>
</table>
Example: Planar 3-link robot

$\begin{align*}
x &= l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y &= l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\
\phi &= \theta_1 + \theta_2 + \theta_3
\end{align*}$

Take $l_1 > l_2 > l_3$, $l_1 > l_2 + l_3$

What is the reachable space?
Take $l_1, l_2$ fixed and vary $\theta_3$

Now vary $\theta_1$

Finally, vary $\theta_2$
Example (continued)

What is the dextrous workspace in the example?
The IK Problem

• Kinematic Problem: given joint angles and/or displacement, compute location and orientation of End Effector.

• Inverse Kinematic Problem: given location and orientation of EE, find joint variables.

• Why is IK hard?
  – May have more than one solution or none at all
  – Amounts to solving nonlinear transcendental equations (can be hard)
Existence of Solutions

- A solution to the IKP exists if the target belongs to the workspace.
- Workspace computation may be hard. In practice is made easy by special design of the robot.
- The IKP may have more than one solution. How to choose the appropriate one?
Multiple Solutions

• Pick the closest solution
  – In joint space
  – Small vs large joints (put weights)
  – Obstacles?
<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>$-90^\circ$</td>
<td>0</td>
<td>0</td>
<td>$\theta_2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$a_2$</td>
<td>$d_3$</td>
<td>$\theta_3$</td>
</tr>
<tr>
<td>4</td>
<td>$-90^\circ$</td>
<td>$a_3$</td>
<td>$d_4$</td>
<td>$\theta_4$</td>
</tr>
<tr>
<td>5</td>
<td>$90^\circ$</td>
<td>0</td>
<td>0</td>
<td>$\theta_5$</td>
</tr>
<tr>
<td>6</td>
<td>$-90^\circ$</td>
<td>0</td>
<td>0</td>
<td>$\theta_6$</td>
</tr>
</tbody>
</table>
For each solution there is another solution in which the last three joints flip to an alternative configuration (O4+180, -O5, O6+180)
# of solutions vs. nonzero $a_i$

- The more that are nonzero – the bigger is the max # of solutions
- Completely general rotary jointed manipulator with 6 DOF -> up to 16 solutions

<table>
<thead>
<tr>
<th>$a_i$</th>
<th>Number of solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 = a_3 = a_5 = 0$</td>
<td>$\leq 4$</td>
</tr>
<tr>
<td>$a_3 = a_5 = 0$</td>
<td>$\leq 8$</td>
</tr>
<tr>
<td>$a_3 = 0$</td>
<td>$\leq 16$</td>
</tr>
<tr>
<td>All $a_i \neq 0$</td>
<td>$\leq 16$</td>
</tr>
</tbody>
</table>
Methods of Solutions

• A manipulator is **solvable** if the joint variables can be determined by an algorithm. The algorithm should find all possible solutions.

• Solutions
  
  \[
  \text{closed form solutions} \\
  \text{numerical solutions}
  \]

We are interested in closed-form solutions

1. Algebraic Methods
2. Geometric Methods
Method of Solution (cont.)

- **Major result:** all systems with revolute and prismatic joints having a total of six degrees of freedom in a single series chain are solvable.

- In general, solution is numerical.

- Robots with analytic solution: several intersecting joint axes and/or many $\alpha_i = 0, 90^\circ$.

- Sufficient condition for 6 revolute joints to have a closed form solution – 3 neighboring joint axes intersect in a point.

- One major application (and driving force) of IK: animation.
Manipulator Subspace when \( n<6 \)

- If \( n<6 \), then the workspace will be a portion of an \( n \) dimensional subspace
- To describe the WS: compute direct kinematics, and then vary joint variables
- On the previous example, the WS has the form:

\[
\begin{bmatrix}
c_{\phi} & -s_{\phi} & 0 & x \\
s_{\phi} & c_{\phi} & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Manipulator Subspace \((n<6)\)

\[
^0 P_{2OOG} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix},
\]
\[ 0 \hat{Z}_2 = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} \\
\frac{y}{\sqrt{x^2 + y^2}} \\
0 \end{bmatrix}. \]

\[ 0 \hat{T}_2 = \begin{bmatrix} \frac{y}{\sqrt{x^2 + y^2}} & 0 & \frac{x}{\sqrt{x^2 + y^2}} & x \\
\frac{-x}{\sqrt{x^2 + y^2}} & 0 & \frac{y}{\sqrt{x^2 + y^2}} & y \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}. \]
Manipulator SuS when n<6 (cont)

• Usual goal for manipulator with $n$ DoF: use $n$ parameters to specify the goal

• If 6 DoF are used, $n<6$ will in general not suffice

• Possible compromise: reach the goal as “near” as possible to original goal:
  - 1) Given the goal frame $\mathbf{T}$, compute modified goal frame $\mathbf{T}'$ in manipulator SuS as near as possible to $\mathbf{T}$
  - 2) Compute IK. A solution may still not be possible if goal is not in the manipulator workspace

• For example, place tool frame origin at desired location, then select a feasible orientation
Algebraic Solution
Algebraic Solution

The kinematics of the example seen before are:

\[
\begin{bmatrix}
  c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} \\
  s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

Assume goal point is specified by 3 numbers:
Algebraic Solution (cont.)

By comparison, we get the four equations:

\[ c_\phi = c_{123} \quad s_\phi = s_{123} \]
\[ x = l_1 c_1 + l_2 c_{12} \]
\[ y = l_1 s_1 + l_2 s_{12} \]

Summing the square of the last 2 equations:

\[ x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 c_2 \]

From here we get an expression for \( c_2 \)

\[ c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2} \]
Algebraic Solution (III)

• When does a solution exist?
• What is the physical meaning if no solution exists?
• Two solutions for $\theta_2$ are possible. Why?

Using $c_{12}=c_1c_2-s_1s_2$ and $s_{12}=c_1s_2-c_2s_1$:

\[
\begin{align*}
x &= k_1 c_1 - k_2 s_1 \\
y &= k_1 s_1 + k_2 c_1
\end{align*}
\]

where $k_1=l_1+l_2c_2$ and $k_2=l_2s_2$. To solve these eqs, set $r=+\sqrt{k_1^2+k_2^2}$ and $\gamma=\text{Atan2}(k_2,k_1)$. 
Then: $k_1 = r \cos \gamma$, $k_1 = r \sin \gamma$, and we can write:

\[ x/r = \cos \gamma \cos \theta_1 - \sin \gamma \sin \theta_1 \]
\[ y/r = \cos \gamma \sin \theta_1 + \sin \gamma \cos \theta_1 \]

or: $\cos(\gamma + \theta_1) = x/r$, $\sin(\gamma + \theta_1) = y/r$
Therefore:
\[ \gamma + \theta_1 = \text{Atan2}(y/r, x/r) = \text{Atan2}(y, x) \]
and so:
\[ \theta_1 = \text{Atan2}(y, x) - \text{Atan2}(k_2, k_1) \]
Finally, \( \theta_3 \) can be solved from:
\[ \theta_1 + \theta_2 + \theta_3 = \phi \]
Geometric Solution

IDEA: Decompose spatial geometry into several plane geometry problems

Applying the “law of cosines”:

\[ x^2 + y^2 = l_1^2 + l_2^2 - 2l_1l_2\cos(180 - \theta_2) \]
Geometric Solution (II)

Then:

\[
c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}
\]

The LoC gives:

\[
l_2^2 = x^2 + y^2 + l_1^2 - 2l_1 \sqrt{x^2 + y^2} \cos \psi
\]

So that \( \cos \psi = \frac{(x^2 + y^2 + l_1^2 - l_2^2)}{2l_1 \sqrt{x^2 + y^2}} \)

We can solve for \( 0 \leq \psi \leq 180 \), and then \( \theta_1 = \beta \pm \psi \)
Reduction to Polynomial

• Trascendental equations difficult to solve since one variable $\theta$ usually appears as $\cos \theta$ and $\sin \theta$.

• Can reduce to polynomial in variable $u = \tan \theta/2$

by using:

\[ \cos \theta = \frac{(1-u^2)}{(1+u^2)} \]
\[ \sin \theta = \frac{2u}{(1+u^2)} \]

How? Use the fact that $\sin \theta = 2[\sin(\theta/2)\cos(\theta/2)]$ and $\cos \theta = [\cos(\theta/2)^2 - \sin(\theta/2)^2]$
Piper’s Solution – 3 axis intersect

- Solution for manipulators with 6DOF’s when three consecutive axis intersect
- We will consider the case of revolute joints and last three axis intersect
- Recall the transformation:

\[
\begin{bmatrix}
  c\theta_i & -s\theta_i & 0 & a_{i-1} \\
  s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\
  s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]
Piper’s Solution

\[ 0P_{4\text{ORG}} = 0T_1T_2T_3T_4P_{4\text{ORG}} = [x \ y \ z \ 1]^T \]

The vector \( P_{4\text{ORG}} \) in the 3-frame has the form:

\[ 3P_{4\text{ORG}} = \begin{bmatrix} a_3 \\ -s\alpha_3d_4 \\ c\alpha_3d_4 \\ 1 \end{bmatrix} \]

And in the 3-frame:

\[ 2P_{4\text{ORG}} = 2T_3 \begin{bmatrix} a_3 \\ -s\alpha_3d_4 \\ c\alpha_3d_4 \\ 1 \end{bmatrix} = \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = \begin{bmatrix} a_3c_3 + d_4s\alpha_3s_3 + a_2 \\ a_3c\alpha_2s_3 - d_4s\alpha_3c\alpha_2c_3 - d_4s\alpha_2c\alpha_3 - d_3s\alpha_2 \\ a_3s\alpha_2s_3 - d_4s\alpha_3s\alpha_2c_3 + d_4c\alpha_2c\alpha_3 + d_3c\alpha_2 \\ 1 \end{bmatrix} \]
Piper’s Solution (II)

Repeating patiently

\[
0_{\mathbf{P}_{4\mathbf{ORG}}} = \begin{bmatrix}
  c_1 g_1 - s_1 g_2 \\
  s_1 g_1 + c_1 g_2 \\
  g_3 \\
  1
\end{bmatrix}
\]

\[
g_1 = c_2 f_1 - s_2 f_2 + a_1
\]
\[
g_2 = s_2 c_1 f_1 + c_2 c_1 f_2 - s_1 f_3 - d_2 s_1 \\
\]
\[
g_3 = s_2 s_1 f_1 + c_2 c_1 f_2 + c_1 f_3 + d_2 c_1
\]

As mentioned before, for rotational joint

\[
\|0_{\mathbf{P}_{4\mathbf{ORG}}}\|
\]

does not depend on \( \theta_1 \):

\[
r = \|0_{\mathbf{P}_{4\mathbf{ORG}}}\|^2 = x^2 + y^2 + z^2 = g_1^2 + g_2^2 + g_3^2
\]
Replace the $g_i$’s, using $z = g_3$ and work patiently:

$$r = (k_1c_2 + k_2s_2)2a_1 + k_3$$
$$z = (k_1s_2 - k_2c_2)s\alpha_1 + k_4$$

$$k_1 = f_1$$
$$k_2 = -f_2$$

$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3$$

$$k_4 = f_3c\alpha_1 + d_2c\alpha_1$$

1. If $a_1=0$, $r = k_3(\theta_3)$. Solve for $\theta_3$

2. If $\sin(\alpha_1)=0$, $z = k_4(\theta_3)$. Solve for $\theta_3$

3. Otherwise, eliminate $s_2$ and $c_2$ above to get

$$\frac{(r-k_3)^2}{4a_1^2} + \frac{(z-k_4)^2}{s\alpha_1^2} = k_1^2 + k_2^2$$
Piper’s Solution (IV)

- 1 and 2 above give a quadratic equation in \( \tan(\theta_3/2) \)
- 3 gives an equation of degree four
- Having solve \( \theta_3 \), can solve above for \( \theta_2 \) and \( \theta_1 \)
- The remaining angles can be computed to give the desired orientation.
Forward Kinematics

• position equation:

\[
0^P = 0^e_T e^P
\]

\[
0^e_T = 0^1_T 1^2_T \cdots N^{-1}_T N^e_T
\]

• Where the transformation for each link is built from our DH parameters:

\[
i^{-1}_{i-1} T = \begin{bmatrix}
c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\
s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & a_i s\theta_i \\
0 & s\alpha_i & c\alpha_i & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Forward Kinematics

• Forward Kinematic Transformation:

\[ e^T_0 = \begin{bmatrix} e^R_0 & e^t_0 \\ 0 & 1 \end{bmatrix} \quad e^t_0 = \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} \]

• Has an rotation and translation:

\[ e^T_0 = \text{Trans} [ e^t_0 ] \text{RPY}[\phi, \theta, \psi] \quad \text{Roll,pitch,yaw} \]

\[ e^T_0 = \text{Trans} [ e^t_0 ] \text{Rot}[\vec{k}, \theta] \quad \text{Angle-axis} \]

\[ e^T_0 = \text{Trans} [ e^t_0 ] R_{ZYX}[\phi, \theta, \psi] \quad \text{Euler Y-Z-Y} \]
Forward Kinematics

• Pick one rotation description:

\[ ^0_eT = \text{Trans}[^0_e\mathbf{t}]\text{RPY}[\phi, \theta, \psi] \]

\[ ^0_e\mathbf{R} = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix} \]

• Decompose \( R \), i.e. solve:

\[ \begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix} = \text{RPY}[\phi, \theta, \psi] \]
Forward Kinematics

- Pick one rotation description:

\[
\begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{bmatrix} = \\
\begin{bmatrix}
C_\phi C_\theta & C_\phi S_\theta S_\psi - S_\phi C_\psi & C_\phi S_\theta C_\psi + S_\phi S_\psi \\
S_\phi C_\theta & S_\phi S_\theta S_\psi + C_\phi C_\psi & S_\phi S_\theta C_\psi - C_\phi S_\psi \\
-S_\theta & C_\theta S_\psi & C_\theta C_\psi
\end{bmatrix}
\]

Find terms that you can easily solve equations
Decomposition of Rotation Matrices

- Roll, Pitch, and Yaw \( \text{Rot}[z, \phi]\text{Rot}[y, \theta]\text{Rot}[x, \psi] \)

\[
\begin{bmatrix}
C\phi C\theta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi \\
S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi \\
-S\theta & C\theta S\psi & C\theta C\psi
\end{bmatrix}
\]

- The inverse solution for \( \theta \) in \((-\pi/2, \pi/2)\):

\[
\phi = \text{atan2}(r_{21}, r_{11})
\]
\[
\theta = \text{atan2}(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2})
\]
\[
\psi = \text{atan2}(r_{32}, r_{33})
\]
Decomposition of Rotation Matrices

- Roll, Pitch, and Yaw

\[
\begin{bmatrix}
C\phi C\theta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi \\
S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi \\
-S\theta & C\theta S\psi & C\theta C\psi \\
\end{bmatrix}
\]

- The inverse solution for \( \theta \) in \((\pi/2, 3\pi/2)\):

\[
\phi = atan2(-r_{21}, -r_{11})
\]
\[
\theta = atan2(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2})
\]
\[
\psi = atan2(-r_{32}, -r_{33})
\]
A Heuristic for Inverse Kinematics

• The solution to the inverse kinematic equations (called the arm solution) can often be found using a heuristic approach.
  – Does not guarantee a solution
  – Solution may not be unique
  – Some solutions may be redundant
IK Heuristic Algorithm

1. Perform forward kinematics to find the general transformation matrix

\[ ^0_eT(q_1, q_2, \ldots, q_N) \]

2. Equate the transformation matrix to the manipulator transformation matrix.
   a) For a particular solution the manipulator transformation matrix contains “numbers”
   b) For a general solution the manipulator

\[ ^0_eT(q_1, q_2, \ldots, q_N) = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
3. Look at both matrices for:
   a) Elements which contain only one joint variable
   b) Pairs of elements which will produce an expression in only one joint variable when divided (look for divisions that result in the \( \text{atan2} \) function)
   c) Elements, or combinations of elements, that can be simplified using trigonometric identities

4. Having selected an element, equate it to the corresponding element in the other matrix to produce an equation. Solve this equation to find a description of one joint variable in terms of the elements of the manipulator transformation matrix.
IK Heuristic Algorithm

5. Repeat step 4 until all the elements identified in step 3 have been used.

6. If any of these solutions suffer from inaccuracies, undefined results, or redundant results, set them aside and look for better solutions.
7. If there are more joint angles to be found, pre-multiply both sides of the matrix equation by the inverse of the $A_1$ matrix to produce a new set of equivalent matrix elements.

\[
\begin{array}{cccc}
\begin{bmatrix}
0 & 1 & T & \cdots & N & -1 & T & N & T
\end{bmatrix}
& = & \begin{bmatrix}
r_{11} & r_{12} & r_{13} & t_1 \\
r_{21} & r_{22} & r_{23} & t_2 \\
r_{31} & r_{32} & r_{33} & t_3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
\]

\[
\begin{array}{cccc}
\begin{bmatrix}
1 & 2 & T & \cdots & N & -1 & T & N & T
\end{bmatrix}
& = & \begin{bmatrix}
r_{11} & r_{12} & r_{13} & t_1 \\
r_{21} & r_{22} & r_{23} & t_2 \\
r_{31} & r_{32} & r_{33} & t_3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
\]
IK Heuristic Algorithm

8. Repeat steps 3 to 7 until either solutions to all the joint variables have been found, or you have run out of A matrices to pre-multiply.

9. If a suitable solution cannot be found for a joint variable, choose one of those discarded in step 6, taking note of regions where problems may occur.

10. If a solution cannot be found for a joint variable in terms of the elements of the manipulator transform, it may be that the manipulator cannot achieve the specified position and orientation: the position is outside the manipulator’s workspace.
IK Heuristic Algorithm

• Note: theoretical solutions may not be physically attainable because of the mechanical limits on the range of joint variables.
Example: PUMA560

- Want to solve:

\[
_{0}^{6} T = \begin{bmatrix}
  r_{11} & r_{12} & r_{13} & p_x \\
  r_{21} & r_{22} & r_{22} & p_y \\
  r_{31} & r_{32} & r_{33} & p_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
  0_{1} T(\theta_1) & 0_{2} T(\theta_2) & 0_{3} T(\theta_3) & 0_{4} T(\theta_4) & 0_{5} T(\theta_5) & 0_{6} T(\theta_6)
\end{bmatrix}
\]

- TRICK: Invert transformations to separate variables:

\[
\left(0_{1} T(\theta_1)\right)^{-1} _{0}^{6} T = \begin{bmatrix}
 0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
 0_{2} T(\theta_2) & 0_{3} T(\theta_3) & 0_{4} T(\theta_4) & 0_{5} T(\theta_5) & 0_{6} T(\theta_6)
\end{bmatrix}
\]
PUMA560: DK Solution

\[
\begin{bmatrix}
1r_{11} & 1r_{12} & 1r_{13} & 1p_x \\
1r_{21} & 1r_{22} & 1r_{23} & 1p_y \\
1r_{31} & 1r_{32} & 1r_{33} & 1p_z \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\begin{align*}
1r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\
1r_{21} &= -s_4c_5c_6 - c_4s_6, \\
1r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\
1r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\
1r_{22} &= s_4c_5s_6 - c_4c_6, \\
1r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\
1r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\
1r_{23} &= s_4s_5, \\
1r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
1p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\
1p_y &= d_3, \\
1p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{align*}
• Then:

\[
\begin{bmatrix}
  c_1 & s_1 & 0 & 0 \\
- s_1 & c_1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} & p_x \\
  r_{21} & r_{22} & r_{22} & p_y \\
  r_{31} & r_{32} & r_{33} & p_z \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}
= t_6 \mathbf{T}(\theta_2 \cdots \theta_6)
\]

• Equating (2,4) element from two sides:

\[-s_1 p_x + c_1 p_y = d_3\]

• Equation can also be obtained from “geometrical” arguments
• Two possible solutions
Make the trigonometric substitutions:

\[ p_x = \rho \cos \phi, \]
\[ p_y = \rho \sin \phi, \]

where

\[ \rho = \sqrt{p_x^2 + p_y^2}, \]
\[ \phi = \text{Atan2}(p_y, p_x) \]
\[ c_1 s_\phi - s_1 c_\phi = \frac{d_3}{\rho}. \]

Diff of angles:

\[ \sin(\phi - \theta_1) = \frac{d_3}{\rho}, \quad \cos(\phi - \theta_1) = \pm \sqrt{1 - \frac{d_3^2}{\rho^2}}, \]

and so

\[ \phi - \theta_1 = \text{Atan2} \left( \frac{d_3}{\rho}, \pm \sqrt{1 - \frac{d_3^2}{\rho^2}} \right). \]
\[ \theta_1 = \text{Atan2}(p_y, p_x) - \text{Atan2}\left(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}\right) \]

Left side is known, equate elements (1,4) and (3,4)

\[ \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{6} T; \]

\[ c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2, \]
\[ -p_x = a_3 s_{23} + d_4 c_{23} + a_2 s_2. \]

Square those and this one

\[ -s_1 p_x + c_1 p_y = d_3 \]

and sum to obtain -
The size of the translation in \{1\} is independent of \(\theta_1\):

\[
a_3c_3 - d_4s_3 = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2} = K
\]

• Geometrical meaning?
• Two possible solutions
• Write:

\[
\begin{pmatrix}
0 & T(\theta_1) & 1 & T(\theta_2) & 2 & T(\theta_3) \\
1 & 0 & T(\theta_4) & 4 & T(\theta_5) & 5 \\
2 & 3 & 0 & 6 & T(\theta_6) \\
3 & 4 & 5 & 0 & T(\theta_1) & 1 \\
4 & 5 & 6 & 1 & 0 & T(\theta_2) \\
5 & 6 & 1 & 2 & 3 & 0
\end{pmatrix}^{-1}
= \begin{pmatrix}
\frac{3}{4}T(\theta_4) & \frac{4}{5}T(\theta_5) & \frac{5}{6}T(\theta_6) \\
\frac{4}{5}T(\theta_5) & \frac{5}{6}T(\theta_6) & \frac{3}{4}T(\theta_4) \\
\frac{5}{6}T(\theta_6) & \frac{3}{4}T(\theta_4) & \frac{4}{5}T(\theta_5)
\end{pmatrix}
\]

• Repeat now a similar procedure