Tutorial 2: Two-Dimensional Linear Systems

Many of the fundamental concepts of image processing are based on one-dimensional signal processing. In this tutorial, we will expand familiar concepts regarding linear systems from one-dimensional signal processing to two dimensions. Analyzing linear systems and their properties is of great importance for understanding a given system.

Introduction

Consider a system with input $f(x, y)$ and output $g(x, y)$:

$$f(x, y) \rightarrow H \rightarrow g(x, y).$$

- **Linearity:**
  A system $H$ will be defined as linear if for every pair of signals $f_1(x, y), f_2(x, y)$ and two parameters $\alpha, \beta$:
  $$H\{\alpha f_1(x, y) + \beta f_2(x, y)\} = \alpha H\{f_1(x, y)\} + \beta H\{f_2(x, y)\}$$

- **Space invariance:**
  A system $H$ will be defined as spatially-invariant (shift invariant) if a spatial shift of the input signal by $(x_0, y_0)$, results in an identical shift at the output. Meaning, the system $H\{f(x, y)\} = g(x, y)$ is space invariant if
  $$H\{f(x-x_0, y-y_0)\} = g(x-x_0, y-y_0).$$

Question 1

A. Consider the following system:

$$f(x, y) \rightarrow H \rightarrow g(x, y)$$

$$g(x, y) = \begin{cases} 2f(x, y) & , |x| \leq b, |y| \leq b \\ f(x, y) & , else \end{cases}$$

The parameter $b$ is positive.

1. Is the system linear?
2. Is the system space-invariant?
Solution
1. The system is linear since
\[
H \{ \alpha f_1(x, y) + \beta f_2(x, y) \} = \begin{cases} 
2 \left[ \alpha f_1(x, y) + \beta f_2(x, y) \right], & |x| \leq b, |y| \leq b \\
\alpha f_1(x, y) + \beta f_2(x, y), & \text{else}
\end{cases}
\]
\[
= \alpha \begin{cases} 
2 f_1(x, y), & |x| \leq b, |y| \leq b \\
f_1(x, y), & \text{else}
\end{cases}
+ \beta \begin{cases} 
2 f_2(x, y), & |x| \leq b, |y| \leq b \\
f_2(x, y), & \text{else}
\end{cases}
\]
\[
= \alpha H \{ f_1 \} + \beta H \{ f_2 \}
\]
2. The system is spatially-varying (intuitively one can see that the system responds differently based on the input location).
A good example is the input signal:
\[
f(x, y) = \begin{cases} 
1, & |x| \leq b, |y| \leq b \\
0, & \text{else}
\end{cases}
\]
with shift \((x_0, y_0) = (2b, 0)\).
Applying the system to the shifted input yields the output:
\[
f(x-x_0, y-y_0) = f(x-2b, y) = \begin{cases} 
1, & b \leq x \leq 3b, |y| \leq b \\
0, & \text{else}
\end{cases}
\]
\[
H \{ f(x-x_0, y-y_0) \} = \begin{cases} 
1, & b \leq x \leq 3b, |y| \leq b \\
0, & \text{else}
\end{cases}
\]
On the other hand, shifting the output of the system results in
\[
g(x, y) = H \{ f(x, y) \} = \begin{cases} 
2, & |x| \leq b, |y| \leq b \\
0, & \text{else}
\end{cases}
\]
\[
g(x-x_0, y-y_0) = g(x-2b, y) = \begin{cases} 
2, & b \leq x \leq 3b, |y| \leq b \\
0, & \text{else}
\end{cases}
\]