Home Assignment #2
Differential Geometry, Curve Evolution
Submission date: 25/12/2018

• For implementation exercises, hand in all relevant .m files, and relevant images as .png/.gif files.

• Please send the solution by e-mail to gautamppai89@gmail.com

• Solutions should be electronically submitted (doc / pdf / ps for the dry part).

• Solitary hand in - no couples.

Problem 1

1. Let \( C(p) = (x(p), y(p), z(p)) \) be a smooth regular curve with arbitrary parameterization. Let \( \times: \) denote the cross product between two vectors. Prove the following formula for the Euclidean curvature \( \kappa \) of \( C(p) \)

\[
\kappa(p) = \frac{\|C_p \times C_{pp}\|}{\|C_p\|^3},
\]

2. Let us be given the curve

\[
C(p) = \begin{cases} 
(p, 0, e^{-1/p^2}) & p > 0 \\
(p, e^{-1/p^2}, 0) & p < 0 \\
(0, 0, 0) & p = 0 
\end{cases}
\]

Show that \( C(p) \) is regular for all \( p \), and that \( \kappa(p) \neq 0 \) for \( p \neq 0, p \neq \pm\sqrt{2/3} \) and \( \kappa(0) = 0 \).
Problem 2

Let \( C(p) = [x(p), y(p)] \) be a smooth closed curve.

1. Choose a non trivial curve \( C(p) \) and indicate its function. Approximate the curve using enough sample points, draw it using Matlab and show the result.

2. Implement the constant speed curve evolution \( C_t = \overrightarrow{N} \) and draw evolution curves in several different times. Describe the effects of your implementation after several iterations (Is it stable? or does it break after a point? Show plots of the evolution just before and after such an event if applicable)

3. Implement the curvature flow \( C_t = \kappa N \) and draw evolution curves in several different times. Repeat the same exercise as (2.)

4. Apply an equi-affine transformation to the curve and indicate the transformation. Implement the curve evolution \( C_t = |\kappa|^\frac{1}{3} N \) to both the curve and its transformation, and draw the evolution curves in several different times. Empirically show that the flow is indeed invariant to equiaffine transformations of a curve.

Problem 3

We learned in class that the first fundamental form can be used to compute areas and lengths on surfaces.

1. The curve 
   \[ c(t) = [0, t, \sqrt{1 - t^2}] \quad t \in [0, 1] \]
   is a part of the surface parameterized by:
   \[ \psi(u, v) = [u, v, \sqrt{1 - u^2 - v^2}] \quad (u, v) \in \text{Unit Disc in} \; \mathbb{R}^2 \]
   Compute the length of the curve using the first fundamental form.
2. Denote the first fundamental form by:

\[ G = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \]

Prove that the differential area element is given by \( dA = \sqrt{\det(G)} \)

3. Compute the area of the surface from part 1 using the result from part 2.

Problem 4

In this question, we will visualize Gaussian and mean curvatures of a surface. Let us be given a surface \( S \) parametrized by

\[ X(u, v) = [x(u, v), y(u, v), z(u, v)] \]

where \((u, v) \in U \subset \mathbb{R}^2\). Assume the parametrization to be regular and smooth.

1. Load from the file `face.mat` the facial surface. The matrices \( X, Y \) and \( Z \) represent the values of \( x, y \) and \( z \). Assume that the surface is parametrized on the unit square, i.e. \( (u, v) \in [0, 1] \times [0, 1] \). The matrix \( \text{MASK} \) is used for data display (it defines the "region of interest", surface points that we do not want to display correspond to \( \text{NaN} \) values in \( \text{MASK} \)). Plot the surface using the following code:

```matlab
h = surf(X,Y,Z.*MASK);
axis image, shading interp, view([0,90]), axis off;
lighting phong, camlight head,
set(h,'FaceColor',[1,1,1]*0.9, 'EdgeColor', 'none' ... 'SpecularColorReflectance', 0.1, 'SpecularExponent', 100);
```
2. Compute two matrices K representing the Gaussian curvature and H representing the mean curvature, at every point of the surface (use the results we derived in class). Compute the derivatives numerically. Plot the surface with each point colored according to its curvature, using the following code:

```matlab
h = surf(X,Y,Z.*MASK,K);
axis image, shading interp, view([0,90]), axis off, colormap jet, lighting phong, camlight head,
set(h,'SpecularColorReflectance', 0.1, 'SpecularExponent', 100);
```
Plot two separate plots for K and H

3. Plot the surface where each point is colored in one of three colors according to point classification: planar, elliptic, or hyperbolic.

**Problem 5**

Consider a smooth surface \( M \) and a point \( p \in M \). We would like to approximate the surface around \( p \) using a second order approximation of the form

\[
z(x, y) = a + bx + cy + dxy + ex^2 + fy^2.
\]

We can choose a co-ordinate system so that the point \( p \) is at the origin and \( a = b = c = 0 \)

1. Find the mean and Gaussian curvatures of the second order surface approximation at the origin.

2. Prove that it is possible to choose the coordinate system so that \( t(u, v) = z(x(u), y(v)) = gu^2 + hv^2 \). What are \( \kappa_1 \) and \( \kappa_2 \) in terms of \( g, h \)? What is the geometric meaning of this fact?

**Problem 6**

Demonstrate Cartans theorem. Choose a sufficiently smooth non-trivial closed curve with enough number of points. Compute the invariant curvature \( \kappa \) and invariant arclength \( s \) and numerically compute the the differential
invariants \( \{\kappa, \kappa_s\} \). Transform the curve and recompute \( \{\kappa, \kappa_s\} \) and see if the signatures are indeed invariant. **Implement for both Euclidean as well as Equiaffine cases.** Report on numerical difficulties. Report also on scenarios where part of the curve is blocked or occluded. Do the signatures for the un-occluded parts change?

Demonstrate your experiment as follows: Show pairs of plots. The first one is just a plot of your curve in 2D. Corresponding to it show the Cartan’s Signature plot for the curve \( \{\kappa, \kappa_s\} \). Now transform the curve and generate another pair. Your report should contain such plots and a short commentary on what you observe.