Home Assignment #1
The Heat Equation, Calculus of Variations
Submission date: 22/11/2017

- For implementation exercises, hand in all relevant .m files, and relevant images as .png/.gif files.
- Please send the solution by e-mail to cs236861@gmail.com.
- Solutions should be electronically submitted (doc / pdf / ps for the dry part).
- Solitary hand in - no couples.

Problem 1

Exercises 6,10,11,12 from Chapter 1 of the book:
(Note: in Ex. 10 you must prove that your solution works. You may work in a fashion similar to Ex. 8 in the book, but you cannot assume the solution to Ex. 8 is given)

Exercise 6
We saw that the length element of a function $y(x)$ is given by $ds = \sqrt{1 + y'^2} dx$. Show that the area element of a function $z(x,y)$ is given by $da = \sqrt{(1 + z_x^2 + z_y^2)} dxdy$. Next, compute the EL equations for the functional $\int da$. (This is Joseph-Louis Lagrange’s result from 1788).

Exercise 10
What is the kernel for the solution of the general linear heat equation for a signal in an arbitrary dimension $u(x_1, x_2, \ldots, x_N; t)$?

Formally, we say that $u : \mathbb{R}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}$, where $\mathbb{R}^N$ is the $N$ dimensional Euclidean space spanned by the $x_i$ arguments, and $\mathbb{R}_+$ is the ray of positive values that describes time. The arbitrary dimension linear heat equations is $u_t = \Delta u$, where $\Delta u = u_{x_1 x_1} + u_{x_2 x_2} + \ldots + u_{x_N x_N}$ is the Laplacian.
Exercise 11
What is the kernel of the 2D linear affine heat equation
\[
    u_t = \text{div} \left( M \nabla u \right)
\]
where \( M \) is a positive definite, symmetric matrix
\[
    M = \begin{pmatrix} a & c \\ c & b \end{pmatrix}
\]
Plot the level sets (also known as iso-contours, or equal-height contours) of the kernel.

Exercise 12
Apply the 2D linear heat equations \( u_t = u_{xx} + u_{yy} \) to gray-level image \( u(x, y, 0) = I(x, y) \). Program either the explicit form with forward Euler numerical iterations in time and central derivative approximation in space, or by a direct multiplication in the frequency domain, by multiplying the transformed signal with the transformed Gaussian kernel. What is the transform of the Gaussian kernel?

For the explicit form, denote \( U_{i,j}^m = u(idx, jdy, ndt) \); then the iterative update scheme is given formally by
\[
    D_+ U_{i,j}^n = (D_+^x D_+^x + D_+^y D_+^y) U_{i,j}^n
\]
Where the definitions of backward and forward derivatives will be given in Chapter 5, or explicitly
\[
    U_{i,j}^{n+1} = U_{i,j}^n + dt \left( \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{dx^2} + \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{dy^2} \right)
\]
Note: Hand in as an example the cameraman image, for 3 different scales/times.
Problem 2

• Find a critical function \( u \) for the problems
  
  - The functional \( L = \int_0^1 [3(u')^2 + 2u + 2x] \, dx, \, u(0) = u(1) = 1 \).
  
  - The functional \( L = \int_0^1 u\sqrt{1 + (u')^2} \, dx, \, u'(0) = 0, u(1) = 2 \).
  
  - The functional \( L = \int_a^b [u^2 - (u')^2 + ue^x] \, dx, \, u(a) = u(b) = 1 \).
  
  - The functional \( L = \int_0^1 (e^{-\alpha x} (u^2 + au + bu' + c (u')^2)) \, dx, \, u(0) = 0, u(1) = 1 \), where \( \alpha > 0, c > 0 \).
  
  - The functional \( L = \int_1^2 \frac{(u')^2}{x^3} \, dx, \, u(1) = 0, u(2) = 1 \).

Provide the simplest form for \( u(x) \) that satisfies the EL equation.

• Extract from the functional \( L = \int \int g \left( (u(x) - u(z))^2 \right) \, dz \, dx \) an expression for \( u(x) \). Give as simple-as-possible expression for \( g(s) = 1 - e^{-s} \).

• A car is driving along a single-lane road at speed \( v_0 \). The driver just noticed (at \( t = 0 \)) that there’s a red light \( D \) meters ahead of him. We decide to approximate the discomfort of the passengers at each moment as the squared acceleration of the car \( (u_{tt})^2 \). How should the car move given this information to achieve full halt at the junction, minimizing the total experienced discomfort of the passengers, \( \int_0^T (u_{tt})^2 \, dt \) (\( T \) is the time of arrival at the junction)?

  Use calculus of variations to help the driver, calculating the trajectory for a given \( T \).

Problem 3

Prove the Euler Lagrange equation for the functional

\[
L = \int F(x, u, u_x, u_{xx}) \, dx
\]

Hint: Use integration by parts where needed, in a manner similar to the Euler Lagrange equation with only \( u, u_x \) involved.