Tutorial 2

Calculus of variations II, Gauss-Seidel method

Matan Sela
The Euler-Lagrange equation (reminder)

Given the functional

$$f(u) = \int_{x_0}^{x_1} F(x, u(x), u'(x)) \, dx$$

with $F \in C^3$ and all admissible $u(x)$ having fixed boundary values $u(x_0) = u^0$ and $u(x_1) = u^1$.

An extremum of $f(u)$ satisfies the differential equation

$$F_u - \frac{d}{dx} F_{u'} = 0$$

with the boundary conditions $u(x_0) = u^0$ and $u(x_1) = u^1$. 
Special cases of the E-L equation

If the integrand does not depend on the independent variable $x$,

$$ f(u) = \int_{x_0}^{x_1} F(u(x), u'(x)) \, dx, $$

for a solution of the E-L equation, the first-order differential equation

$$ \frac{d}{dx} \left( F - u' F_{u'} \right) = 0, $$

or

$$ F - u' F_{u'} = \text{const}, $$

must hold (Beltrami identity).
Proof of the Beltrami Identity

Using the full derivative of $F$ by $x$ we obtain

\[ \frac{dF}{dx} = \frac{\partial F}{\partial u} u' + \frac{\partial F}{\partial u'} u'' + \frac{\partial F}{\partial x} \rightarrow \frac{\partial F}{\partial u} u' = \frac{dF}{dx} - \frac{\partial F}{\partial u'} u'' - \frac{\partial F}{\partial x} \]

Multiplying the E-L equation by $u'$ we obtain:

\[ u' \frac{\partial F}{\partial u} - u' \frac{d}{dx} \frac{\partial F}{\partial u'} = 0, \]

or

\[ u' \frac{\partial F}{\partial u} = u' \frac{d}{dx} \frac{\partial F}{\partial u'} \]
Plugging the obtained identity into the E-L equation, we get:

\[
\frac{dF}{dx} - \frac{\partial F}{\partial u'} u'' - \frac{\partial F}{\partial x} - \frac{\partial F}{\partial u} u' = 0 \rightarrow
\]

\[
\frac{dF}{dx} - \frac{\partial F}{\partial u'} u'' - \frac{\partial F}{\partial x} - u' \frac{d}{dx} \frac{\partial F}{\partial u'} = 0 \rightarrow
\]

\[
\frac{\partial F}{\partial x} + \frac{dF}{dx} (F - u' \frac{\partial F}{\partial u'}) = 0
\]

Using the assumption that \(\frac{\partial F}{\partial x} = 0\) results in

\[
\frac{d}{dx} \left( F - u' \frac{\partial F}{\partial u'} \right) = 0
\]
Brachistochrone problem

- Formulation and first attempt to prove was done by Galileo, 1638. *Brachistos* - shortest. *Chronos* - time.

- John Bernoulli – 1696 solved the problem and challenged math world (Acta Eroditorium)

  “Given points A and B in a vertical plane, to find the path AB down which a movable point M must, by virtue of its weight, proceed from A to B in the shortest possible time”

- The solution curve is very known by the geometers (cycloid)

- He gave one year for the mathematicians of the time to solve the problem
Solutions were presented by
- Leibniz (received letter 9/6/1696, sent back solution 16/6/1696)
- Newton - sent an anonymous answer the very night..

Figure: Brachistochrone
Let $ds = \sqrt{1 + y'^2} dx$ be a length element.

Energy conservation

$$\frac{1}{2} m \left( \frac{ds}{dt} \right)^2 - mgy \equiv E_0 = \frac{m}{2} V_0^2 - mgy_0$$

$$\Rightarrow \left( \frac{ds}{dt} \right)^2 = 2g(\alpha + y), \quad \alpha = \frac{V_0^2}{2g} - y_0,$$

Our integral becomes:

$$T = \int \frac{dt}{ds} ds = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2} dx}{\sqrt{2g(\alpha + y)}}$$
Problem 1:

\[
\min \left\{ \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{y + \alpha}} \, dx \right\}
\]

\( F_x = 0 \) then the Euler-Lagrange equation is:

\[
H = y' F_{y'} - F = c
\]

\[
\frac{y'^2}{\sqrt{1 + y'^2} \sqrt{y + \alpha}} - \frac{\sqrt{1 + y'^2}}{\sqrt{y + \alpha}} = -1
\]

\[
= \frac{1}{\sqrt{1 + y'^2} \sqrt{y + \alpha}} = c \quad \rightarrow
\]

\[
\frac{1}{\sqrt{1 + y'^2} \sqrt{y + \alpha}} = \frac{1}{\sqrt{2b}} \quad b > 0
\]
Or, more plainly:

\[(\alpha + y)(1 + y'^2) = 2b\]  \hspace{1cm} (1)

Note:

\[F_y = \frac{\sqrt{1+y'^2}}{(\alpha + y)^{3/2}} \neq 0,\]  \hspace{1cm} (2)

so we cannot take a shortcut..

Note: \(\iff y' \equiv 0\) are not extremals of our functional because of the physical problem we solve (kinetic energy..).

In order to solve (1), let us denote

\[y'(x) = -\tan\frac{\tau}{2}, \quad -\frac{\pi}{2} < \frac{\tau}{2} < \frac{\pi}{2}\,.

\hspace{1cm} \iff \alpha + y = \frac{2b}{1+y'^2}.\]  \hspace{1cm} (3)
\[
\frac{2b}{1 + \tan^2\left(\frac{\tau}{2}\right)} = \frac{2b \cos^2\left(\frac{\tau}{2}\right)}{\cos^2\left(\frac{\tau}{2}\right) + \sin^2\left(\frac{\tau}{2}\right)} = b \ 2 \cos^2\frac{\tau}{2} = b(1 + \cos \tau),
\]

or

\[
y = b(1 + \cos \tau) - \alpha.
\]

\[
\frac{dx}{d\tau} = \frac{dx}{dy} \frac{dy}{d\tau} = -\frac{1}{\tan \frac{\tau}{2}}(-b \sin \tau) = b(1 + \cos \tau)
\]

Integrating Eq. 5, we obtain

\[
\Rightarrow \quad x = a + b(\tau + \sin \tau), \quad -\pi \leq \tau \leq \pi.
\]
Problem 2: (Hyperbolic Geodesics)

\[ \min\int_{1}^{2} \frac{\sqrt{1 + y'^2}}{x} \, dx \]

\[ y(1) = 0 \quad y(2) = 1 \]

\[ F = \frac{\sqrt{1 + y'^2}}{x} \]

F is independent of y, and therefore we use

\[ F_{y'} = c \quad \Rightarrow \quad \frac{y'}{x\sqrt{1 + y'^2}} = c \]
\[ y'' = c^2 x^2 (1 + y''^2) \]
\[ y''^2 (1 - c^2 x^2) = c^2 x^2 \]
\[ y' = \pm \frac{cx}{\sqrt{1 - c^2 x^2}} \]
\[ y = \pm \frac{\sqrt{1 - c^2 x^2}}{c} + c_2 \]
\[ (y - c_2)^2 + x^2 = \frac{1}{c^2} \]

boundary conditions \[ \implies c = \frac{1}{\sqrt{5}} \quad c_2 = 2 \]

And the solution is:

\[ (y - 2)^2 + x^2 = 5 \]
Problem 3: Constrained maximum

Find a curve $y(x)$ with fixed endpoints $y(\pm 1) = 0$ and perimeter

$$S = \int_{-1}^{1} \sqrt{1 + y'^2} \, dx,$$

which maximizes the area

$$A(y) = \int_{-1}^{1} y \, dx.$$

Solution

Construct the Lagrangian

$$L(y) = \int_{-1}^{1} y \, dx + \lambda \left( \int_{-1}^{1} \sqrt{1 + y'^2} \, dx - S \right),$$

and denote

$$F(x, y, y') = y + \lambda \sqrt{1 + y'^2}.$$
Problem 3: Constrained maximum (cont.)

The Euler-Lagrange equation

\[ 0 = \frac{d}{dx} F_{y'} - F_y = \frac{d}{dx} \left( \frac{\lambda y'}{\sqrt{1 + y'^2}} \right) - 1. \]

Integration w.r.t. \( x \) yields

\[ \frac{\lambda y'}{\sqrt{1 + y'^2}} = x - \alpha, \]

where \( \alpha = \text{const.} \). Substitute \( y' = \tan \theta \):

\[ \frac{\lambda y'}{\sqrt{1 + y'^2}} = \lambda \frac{\sin \theta}{\cos \theta} \]

\[ = \lambda \frac{\sin \theta}{\sqrt{\sin^2 \theta + \cos^2 \theta}} \]

\[ = \lambda \frac{\sin \theta}{\cos \theta} \sqrt{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} \]

\[ = \lambda \sin \theta, \]

from where

\[ \sin \theta = \frac{x - \alpha}{\lambda}. \]
Substituting again

\[ y' = \tan \theta = \frac{\sin \theta}{\cos \theta} = \pm \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} \]

\[ = \frac{\pm (x - \alpha)}{\lambda \sqrt{1 - \frac{(x - \alpha)^2}{\lambda^2}}} = \frac{\pm (x - \alpha)}{\sqrt{\lambda^2 - (x - \alpha)^2}}. \]

Integration w.r.t. \( x \) yields (table of integrals or Mathematica)

\[ y = \sqrt{\lambda^2 - (x - \alpha)^2} + \beta, \]

or

\[ (x - \alpha)^2 + (y - \beta)^2 = \lambda^2 \]

where \( \beta = const \). The latter equation describes a part of a circle with radius \( \lambda \) centered at \( (\alpha, \beta) \). The exact values of the constants are determined using the endpoint conditions and the perimeter constraint.
Problem 4

Note about the Beltrami identity: it merely states a necessary condition for an extremum of the functional.

Q: Show that a solution of the Beltrami identity does not have to solve the Euler Lagrange equation.

A: Look at

\[
\int \left( \frac{1}{2}(u')^2 - u \right) \, dx
\]

and the function \( u \equiv 1 \), for which the Beltrami identity holds

\[
\frac{d}{dx} \left( F - u' F_{u'} \right) = \\
\frac{d}{dx} \left( \frac{1}{2}(u')^2 - u - u'(u') \right) = \\
\frac{d}{dx} \left( -\frac{1}{2}(u')^2 - u \right) = -(u')(u'') - u' = 0,
\]
and yet,

\[ F_u - \frac{d}{dx} F_{u'} = \]

\[ -1 \frac{d}{dx}(u') = -1 - u'' = 0 \rightarrow u'' = -1. \]

In fact, according to the E-L equation, extrema of the functional are of the form

\[ u(x) = -x^2 + ax + b \]

This is physically not very surprising – the integrand represents the kinetic energy \( T \) of a mass plus its potential energy \(-V\), in 1D. The resulting functions describe a free-fall behavior, subject to initial position and velocity. The functional is known as the action integral, or action.

*Hamilton’s principle* states that the path of a particle (in a system which conserves the total energy, \( E = T - V \)) must be such that it describes an extremum of the action integral.
A practical example:
Optical flow – Horn and Schunck’s method

Given 2 images, we would like to find a field that translates pixels from one image to the next
In a 1D world, we would look at an image flow field $u(x)$ that moves pixels from $I_1$ to $I_2$. We would like to preserve the brightness between the two images. This is known as the *brightness constancy assumption*. Using 1st order Taylor approximation, we would get:

$$I_1(x + u) \approx I_1(x) + I_{1,x}u.$$ 

Using the brightness constancy assumption, we have

$$I_{1,x}u + I_1 - I_2 \approx 0$$

.. but this is only approximate, and our images are usually 2D..
Optical flow, Horn and Schunck’s method (cont.)

In a 2D image case, we denote the field at each point as $u(x, y), v(x, y)$. Taylor approximation of $I$ now reads:

$$I_1(x + u, y + v) \approx I_1 + I_{1,x}u + I_{1,y}v.$$  

The brightness constancy assumption gives us the following functional on $u, v$:

$$\int_{\Omega} (I_{1,x}u + I_{1,y}v + l_1 - l_2)^2 d\Omega$$

.. but this is not enough! We need some way to propagate information in the field.. Otherwise, the constraint

$$I_xu + I_yv = l_t$$

will not suffice (2 variables, 1 equation).
Since using a single constraint is not enough to determine $u$ and $v$, we therefore add regularization to the functional:

$$
\int_{\Omega} \left[ (l_{1,x}u + l_{1,y}v + l_1 - l_2)^2 + \lambda(|\nabla u|^2 + |\nabla v|^2) \right] d\Omega
$$

Denoting $l_t = l_2 - l_1$, we can write the equation as:

$$
\int_{\Omega} \left[ (l_{x}^2 u^2 + l_{y}^2 v^2 + l_t^2 + 2l_{x}l_{y} uv + 2l_{x}l_t u + 2l_{x}l_t v) + \lambda(u_x^2 + u_y^2 + v_x^2 + v_y^2) \right] d\Omega
$$
Writing the two Euler-Lagrange equations (for $u$ and $v$), we get:

$$
(2l_x^2 u + 2l_x l_y v) + \lambda \left( \frac{d}{dx} u_x + \frac{d}{dy} u_y \right) = -2l_x l_t \\
\Delta u
$$

$$
(2l_x l_y u + 2l_y^2 v) + \lambda \left( \frac{d}{dx} v_x + \frac{d}{dy} v_y \right) = -2l_y l_t \\
\Delta v
$$

We see that this regularization boils down to diffusing the components $u, v$ of the field. This is not a coincidence!
Optical Flow, Horn and Schunck’s method (cont.)

If we add several more tricks, we can obtain a nice result, (after some work...):

Figure: Optical flow (taken from A. Bruhn and J. Weickert, IJCV’05)