236861 Geometric Computer Vision

Tutorial 1

Calculus of variations

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What is calculus of variations?

- Deals with functions of functions – functionals
- Defines optimal functions, optimal surfaces, optimal images, and so forth..
What is *calculus of variations*?

**Calculus**

1. **Function**
   \[ f : \mathbb{R}^n \mapsto \mathbb{R} \]

Example:
\[ f(x, y) = x^2 + y^2 \]

2. **Derivative**
\[ \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \]
\[ df = \frac{\partial}{\partial \epsilon} f(x + \epsilon \Delta x) \bigg|_{\epsilon=0} \]

**Calculus of variations**

**Functional**
\[ f : \mathbb{F} \mapsto \mathbb{R}, \text{ in particular} \]
\[ f(u) = \int_{\Omega} F(x, u(x), u'(x), ...) \, dx \]

Example:
\[ f(u) = \int_{\Omega} \| \nabla u(x, y) \|_2 \, dx \, dy \]

**Variation**
\[ \frac{\delta f(u)}{\delta u} = \lim_{\epsilon \to 0} \frac{f(u + \epsilon \delta u) - f(u)}{\epsilon} \]
\[ \delta f = \frac{\partial}{\partial \epsilon} f(u + \epsilon \delta u) \bigg|_{\epsilon=0} \]
What is calculus of variations?

**Figure:** Variations of a function as high dimensional directions
What is calculus of variations?

Calculus

3. **Local (relative) minimum**
   \[ f(x^*) \leq f(x) \]
   \[ \forall x : \| x - x^* \| \leq \alpha \]

4. **Necessary condition for local extremum**
   \[ \frac{df(x)}{dx} = 0 \]
   \[ \frac{\delta f(u)}{\delta u} = 0 \]

5. **Sufficient condition for local minimum**
   \[ \frac{d^2f(x)}{dx^2} \geq 0 \]
   \[ \nabla^2 f(x) \geq 0 \]

Calculus of variations

3. **Local (relative) minimum**
   \[ f(u^*) \leq f(u) \]
   \[ \forall u : \max_{x \in \Omega} |u(x) - u^*(x)| \leq \alpha \]

4. **Necessary condition for local extremum**
   \[ \frac{df(u)}{du} = 0 \]

5. **Sufficient condition for local minimum**
   \[ \nabla^2 f(u) \geq 0 \]

More complex theory
What is *calculus of variations*?

**Calculus**

3. Constrained local minimum

\[
\min_x f(x) \quad \text{s.t.} \quad g(x) = 0
\]

**Calculus of variations**

\[
\min_{u(x)} \int_{\Omega} F(x, u(x), u'(x)) \, dx \quad \text{s.t.} \quad \int_{\Omega} G(x, u(x), u'(x)) = 0
\]

4. Lagrangian

\[
\ell(x) = f(x) + \lambda g(x)
\]

\[
\ell(u) = \int_{\Omega} (F + \lambda G) \, dx
\]

5. Method of Lagrange multipliers

\[
\frac{d\ell(x^*)}{dx} = 0 \quad \text{s.t.} \quad g(x) = 0
\]

\[
\frac{\delta \ell(u^*)}{\delta u} = 0 \quad \int_{\Omega} G(x, u^*(x), u'^*(x)) = 0
\]
Examples of functionals

1. **Curve length**\( \mathcal{L}(y) = \int_{x_0}^{x_1} \sqrt{1 + y'^2} \, dx \)

*The length of a non-parametric curve \( y(x) \).*

2. **Curve length**\( \mathcal{L}(x, y) = \int_{0}^{1} \sqrt{x'^2 + y'^2} \, dt \)

*The length of a parametric curve \((x(t), y(t))\).*
Examples of functionals

3. **Surface area**  
   \[ A(z) = \int_S \sqrt{1 + z_x^2 + z_y^2} \, dx dy \]

   *The area of a non-parametric surface* \( S = (x, y, z(x, y)) \).
Examples of functionals

4. **Total variation** \( TV(y) = \int_{x_0}^{x_1} |y'| \, dx \)

The “oscillation strength” of a non-parametric curve \( y(x) \).

![Graphs showing original signal, noisy signal, and denoised signal.]

**Figure:** Left to Right: original signal, noisy signal, denoised signal.
The Euler-Lagrange Equation

Given the functional

\[ f(u) = \int_{x_0}^{x_1} F(x, u(x), u'(x)) \, dx \]

with \( F \in C^3 \) and all admissible \( u(x) \) having fixed boundary values \( u(x_0) = u^0 \) and \( u(x_1) = u^1 \).

An extremum of \( f(u) \) satisfies the differential equation

\[ F_u - \frac{d}{dx} F_{u'} = 0 \]

with the boundary conditions \( u(x_0) = u^0 \) and \( u(x_1) = u^1 \).

This equation is known as the Euler-Lagrange equation.
Note: the second term of the equation does not have a parallel in a finite dimension gradient expression from multivariate calculus – this term’s form expresses the relation of altering the value of $u$ based on the effect of its derivative. Later on we will see how this term stems from the derivation of the E-L equation.
The E-L equation (independent on $u(x)$)

If the integrand does not depend on $u(x)$,

$$f(u) = \int_{x_0}^{x_1} F(x, u'(x)) \, dx,$$

the E-L equation becomes a first-order differential equation

$$\frac{d}{dx} F_{u'} = 0$$

or

$$F_{u'} = \text{const}.$$
The E-L equation (independent on $u'(x)$)

If the integrand does not depend on $u'(x)$,

$$f(u) = \int_{x_0}^{x_1} F(x, u(x)) \, dx,$$

the E-L equation becomes an algebraic equation

$$F_u(x, u(x)) = 0.$$
The E-L equation (high-order functionals)

Given the functional

\[ f(u) = \int_{x_0}^{x_1} F(x, u(x), u'(x), \ldots, u^{(n)}(x)) \, dx \]

with \( F \in C^{n+2} \) and fixed boundary values \( u^{(i)}(x_0) = u_0^{(i)} \) and \( u^{(i)}(x_1) = u_1^{(i)} \) for \( i = 0, \ldots, n-1 \).

The Euler-Lagrange equation (also known as the Euler-Poisson equation) is

\[ F_u - \frac{d}{dx} F_{u'} + \frac{d^2}{dx^2} F_{u''} + (-1)^n \frac{d^n}{dx^n} F_{u^{(n)}} = 0. \]
The E-L equation (multiple independent variables)

Given the functional

\[
    f(u) = \int_{\Omega} F(x, u(x), u_{x_1}(x), \ldots, u_{x_n}(x)) \, dx
\]

with \( x \in \mathbb{R}^n \) and \( u|_{\partial\Omega} = u_{\partial\Omega} \).

An extremum of \( f(u) \) satisfies the differential equation

\[
    \frac{\partial F}{\partial u} - \frac{d}{dx_1} \frac{\partial F}{\partial u_{x_1}} - \ldots - \frac{d}{dx_n} \frac{\partial F}{\partial u_{x_n}} = 0
\]

with the boundary condition \( u|_{\partial\Omega} = u_{\partial\Omega} \).
The E-L equation (multiple functions)

Given the functional

\[ f(u) = \int_{x_0}^{x_1} F(x, u_1(x), ..., u_n(x), u'_1(x), ..., u'_n(x)) \, dx \]

with \( F \in C^3 \) and all admissible \( u(x) \) having fixed boundary values \( u_i(x_0) = u_i^0 \) and \( u_i(x_1) = u_i^1 \).

An extremum of \( f(u) \) satisfies the system of differential equations

\[ F_{u_i} - \frac{d}{dx} F_{u'_i} = 0 \]

with the boundary conditions \( u(x_0) = u_0 \) and \( u(x_1) = u_1 \).
Proof of the Euler-Lagrange equation

For an optimal function $u(x)$, we require

$$\frac{d\tilde{f}}{d\epsilon} = 0, \quad \tilde{f}(u) = \int F(x, \tilde{u}, \tilde{u}')dx, \quad \tilde{u}(x) = u(x) + \epsilon\eta(x)$$

Developing the term for the variation we obtain:

$$\frac{d\tilde{f}}{d\epsilon} = \int \frac{dF}{d\epsilon}(x, \tilde{u}, \tilde{u}')dx = \int \left[ \frac{\partial F}{\partial x} \frac{\partial x}{\partial \epsilon} + \frac{\partial F}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial \epsilon} + \frac{\partial F}{\partial \tilde{u}'} \frac{\partial \tilde{u}'}{\partial \epsilon} \right] dx = \int \eta \frac{\partial F}{\partial u} + \eta' \frac{\partial F}{\partial u'} dx$$
Using integration by parts, we obtain: And hence,

\[
\int \eta \frac{\partial F}{\partial u} + \eta' \frac{\partial F}{\partial u'} \, dx = \\
\int \eta \left( \frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} \right) \, dx + \left[ \eta \frac{\partial F}{\partial u'} \right]_{x_0}^{x_1}
\]

Since this equation should take place for every \( \eta \), and assuming the boundary term cancels out, we obtain the E-L equation:

\[
\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} = 0
\]
Problem 1: Minimum distance on a plane

Prove that the family of curves minimizing the distance

\[ L = \int_{-1}^{1} \sqrt{1 + y'^2} \, dx, \quad y(-1) = a, \, y(1) = b \]

in the plane are straight lines.
Problem 1: Minimum distance on a plane (cont.)

Solution

The Euler-Lagrange equation

\[ 0 = \frac{d}{dx} F_{y'} - F_y = \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right). \]

Integration w.r.t. \( x \) yields

\[ \frac{y'}{\sqrt{1 + y'^2}} = \gamma = \text{const.} \]
Substitute \( y' = \tan \theta \):

\[
\frac{y'}{\sqrt{1 + y'^2}} = \frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{\cos \theta} \sqrt{1 + \sin^2 \theta} = \frac{\sin \theta}{\cos \theta} \sqrt{\frac{\cos^2 \theta}{\cos^2 \theta}} = \frac{\sin \theta}{\cos \theta} \sqrt{\frac{\cos^2 \theta}{1}} = \sin \theta,
\]

from where \( \sin \theta = \gamma \). Substituting again yields

\[
y' = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\pm \sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{\pm \gamma}{\sqrt{1 - \gamma^2}} = \alpha = \text{const},
\]

from where

\[
y = \alpha x + \beta,
\]

\( \beta = \text{const} \). The solution describes a line in the plane; the exact values of \( \alpha, \beta \) are determined from the endpoint values.
Problem 2: Heat Equation

We would like to minimize:

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega} (u_x^2 + u_y^2) d\Omega$$

The Euler-Lagrange equation is:

$$EL(u) = \frac{d}{dx} (F_{u_x}) + \frac{d}{dy} (F_{u_y})$$

$$= \frac{d}{dx} (2u_x) + \frac{d}{dy} (2u_y)$$

$$= 2 \text{div} (\nabla u)$$

$$= 2\Delta u$$

A gradient descent of the functional can be described as

$$u_t = EL(u) = 2\Delta u$$
Problem 3: Total Variation Denoising

We would like to minimize:

$$\int_{\Omega} \sqrt{\epsilon^2 + u_x^2 + u_y^2} d\Omega$$

The Euler-Lagrange equation is:

$$EL(u) = \frac{d}{dx} \left( \frac{2u_x}{2\sqrt{\epsilon^2 + u_x^2 + u_y^2}} \right) + \frac{d}{dy} \left( \frac{2u_y}{2\sqrt{\epsilon^2 + u_x^2 + u_y^2}} \right)$$

$$= \text{div} \left( \frac{\nabla u}{\sqrt{\epsilon^2 + u_x^2 + u_y^2}} \right)$$

Again, we can use the EL condition as a gradient term, to smooth the image:

$$u_t = \text{div} \left( \frac{\nabla u}{\sqrt{\epsilon^2 + u_x^2 + u_y^2}} \right) \approx \text{div} \left( \frac{\nabla u}{|\nabla u|} \right)$$
Problem 3: Total Variation Denoising (cont.)

Figure: Left to Right: original signal, noisy signal, denoised signal, a simple linear diffusion result.
Problem 3: Total Variation Denoising (cont.)

A similar problem is that of inpainting – here we assume a known portion of the pixels were corrupted and apply the TV flow only to them.

Figure: Left to Right: Inpainting using total variation: before (50% of the pixels removed) and after inpainting.