Geometric Computer Vision

Ron Kimmel

Geometric Image Processing Lab.
CS Dept. Technion

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Farthest Point Sampling
Let $X^n = \{x_i\}_{i=1,...,n}$ be a set of $n$ points in some metric space, for example $x_i \in \mathbb{R}^N$, such that $d(x_i, x_j)$ can be computed. We would like to represent the large set $X^n$ by a subset $Y^k = \{y_j\}_{j=1,...,k}$, where $Y^k \subset X^n$ and $k \ll n$.

Denote the maximal distortion by

$$r^k = \max_{x_i \in X^n} \min_{y_j \in Y^k} d(x_i, y_j).$$

The optimal $Y^k$ should satisfy

$$Y^{k*} = \arg \min_{Y^k \subset X^n} r^k.$$

That is, the $k$-quantization subset of $X^n$ that minimizes the $L_\infty$ norm. By definition, $Y^{k*}$ yields the smallest possible $r^{k*}$.

Finding $Y^{k*}$ is, generally speaking, NP hard.

Can we find a $Y^k \subset X^n$ for which the quantization error would be smaller than $c r^{k*}$ for some positive constant $c$?

Such an approximation is known as $c$-approximation.
Farthest Point Sampling

**Init:** Let \( y_1 = x_i \) for an arbitrary \( x_i \in X_n \), and let \( Y = \{ y_1 \} \).

**loop:** Find the point \( x_f \in X^n \) which is the farthest away from the selected points in \( Y \). That is,

\[
x_f = \max_{x_f \in X^n} \min_{y_j \in Y} d(x_f, y_j).
\]

- Let \( Y = Y \cup \{ x_f \} \)
- If \( |Y| < k \) go to **loop**.

Denote the output as \( Y^{k_{FPS}} \).

Next, we will prove that this simple greedy strategy is a 2-approximation.
Proof sketch.
Let \( v_j \) be the Voronoi cell about \( y_j^* \in Y^{k*} \).
\( v_j \) includes all points in \( X_n \) whose distance to \( y_j^* \) is the smallest.
That is, \( x_i \in v_j \) if \( d(x_i, y_j^*) \leq d(x_i, y_l^*) \) for all \( y_l^* \in Y^{k*} \), such that \( l \neq j \).
Using the pigeon hole principle, the proof shows that either each point in \( Y^{k_{FPS}} \) falls into a different Voronoi cell of \( Y^{k*} \), or there is at least one case in which there are two (or more) points of \( Y^{k_{FPS}} \) that fall into the same Voronoi cell of \( Y^{k*} \). In both cases \( r^k \leq 2r^{k*} \).
**Case 1:** Assume that each $y_j \in Y^{k_{FPS}}$ falls into a different Voronoi cell. That is, at each $v_j$ with representative point $y_j^* \in Y^{k*}$, there is exactly one $y_j$. By definition, $d(y_j, y_j^*) \leq r^{k*}$. Next, for each $x_i \in v_j$ we have $d(x_i, y_j^*) \leq r^{k*}$. Since we deal with a metric space, by the triangle inequality, for each $x_i \in v_j$ we have,

$$d(x_i, y_j) \leq d(x_i, y_j^*) + d(y_j^*, y_i) \leq 2r^{k*}.$$ 

Repeating for all points in $X^n$, each point within its own cell, concludes this part.
Case 2: Assume that at step $l \leq k$ of the FPS procedure, two points $y_m$ and $y_l$, where $m < l$, fall into the same Voronoi cell $v_j$ about $y_j^* \in Y^{k*}$. The maximal distortion of the FPS strategy is $r^l \leq r^m$, for all $m < l$. In fact, the closest point in $Y^{l-1}$ to the newly selected $y_l$ is at distance

$$r^{l-1} = \max_{x_i \in X^n} \min_{y_j \in Y^{l-1}} d(x_i, y_j),$$

which is how the FPS selects its candidates from $X^n \setminus Y^{l-1}$. We need to show that if $y_l$ and $y_m$ both fall into $v_j$, then $r^{l-1} \leq 2r^{k*}$. By construction, $r^{l-1} \leq d(y_l, y_m)$. We thus have that

$$r^k \leq r^{l-1} \leq d(y_l, y_m) \leq d(y_l, y_j^*) + d(y_j^*, y_m) \leq 2r^{k*}$$
Matlab Example

```
N = 256; A = zeros(N);
X = 1+floor(rand(1)*N);
Y = 1+floor(rand(1)*N);
A(X,Y) = 1;
for k = 1:1000,
    figure(1);
    B = bwdist(A);
    imagesc(B'); axis image; hold on;
    mx = max(B(:)); ind = (B == mx);
    [I,J] = find(ind,1);
    A(I,J)=1;
    X = [X; I]; Y = [Y; J];
    if max(size(X))>4,
        [vx,vy] = voronoi(X,Y);
        plot(X,Y,'ro',vx,vy,'k-');
        axis equal; axis([1 N 1 N]);
        hold off;
        drawnow;
    end %if
end
```
Chapter Optimal Basis for Smooth Signals

Optimal Basis for Smooth Signals
We will provide the best representation basis (in terms of truncated representation) for a specific family of signals. Denote the family of functions $\psi : \Omega \to \mathbb{R}$ such that $\|\nabla \psi\|^2 \leq 1$ as $\{\psi_\omega\}$, and let $\{\beta_i\}$ be a basis over the smooth bounded domain $\Omega \subset \mathbb{R}^N$.

Again, let the squared representation error with the first $k$ elements of $\{\beta_i\}$ be defined as

$$E^2_k(\psi, \beta) = \left\| \psi - \sum_{i=1}^{k} \langle \psi, \beta_i \rangle \beta_i \right\|^2.$$

We would like to find the basis $\{\beta_i\}$ that minimizes $E^2_k(\psi, \beta)$ for all $k \geq 1$, for all $\psi \in \{\psi_\omega\}$.

We will prove that the eigenfunctions of the Laplace operator ordered by their corresponding eigenvalues in a non-decreasing order uniquely define the optimal basis.
A Note about Dirichlet, Fourier and Laplace

Let \( \psi(t) : [0, 1) \to \mathbb{R} \) be a signal such that w.l.o.g. \( \psi'(0) = 0 \) and \( \psi'(1) = 0 \), also known as Neumann boundary conditions. The Dirichlet energy of \( \psi(t) \) is given by

\[
\| \nabla \psi(t) \|^2 \equiv \langle \nabla \psi(t), \nabla \psi(t) \rangle \\
= \int_0^1 \frac{d}{dt} \psi(t) \frac{d}{dt} \psi(t) dt \\
= \psi(t) \frac{d}{dt} \psi(t) \bigg|_{t=0}^{1} - \int_0^1 \psi(t) \frac{d^2}{dt^2} \psi(t) dt \\
= \langle \psi(t), -\frac{d^2}{dt^2} \psi(t) \rangle \equiv \langle \psi(t), -\Delta \psi(t) \rangle.
\]

We could further solve for

\[
-\frac{d^2}{dt^2} e_i(t) = \lambda_i e_i(t).
\]

Did we say Fourier?
Let $e = \{e_i\}$ be the orthonormal basis consisting the eigenfunctions of the Laplace operator

$$-\Delta e_i = \lambda_i e_i,$$

where we also assume that the eigenfunctions are equal to zero along the boundary of $\Omega$, i.e. $e_i = 0$ on $\partial \Omega$.

The ordered eigenvalues are $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. 
Warmup

We will also use the fact that

$$
\|\nabla \psi\|^2 \equiv \langle \nabla \psi, \nabla \psi \rangle \\
= \langle \psi, -\Delta \psi \rangle \\
= \left( \sum_{i=1}^{+\infty} \langle \psi, e_i \rangle e_i, \sum_{i=1}^{+\infty} \langle -\Delta \psi, e_i \rangle e_i \right) \\
= \sum_{i=1}^{+\infty} \langle \psi, e_i \rangle \langle -\Delta \psi, e_i \rangle \\
= \sum_{i=1}^{+\infty} \langle \psi, e_i \rangle \langle \psi, -\Delta e_i \rangle \\
= \sum_{i=1}^{+\infty} \langle \psi, e_i \rangle \langle \psi, \lambda_i e_i \rangle \\
= \sum_{i=1}^{+\infty} \lambda_i \langle \psi, e_i \rangle^2
$$
Theorem 1 \( \forall k \geq 1 \) we have, \( \forall \psi \in \{ \psi_\omega \} \)

\[
\mathcal{E}_k^2(\psi, e) = \left\| \psi - \sum_{i=1}^{k} \langle \psi, e_i \rangle e_i \right\|^2 \leq \frac{\|\nabla \psi\|^2}{\lambda_{k+1}},
\]

Proof. One one hand,

\[
\left\| \psi - \sum_{i=1}^{k} \langle \psi, e_i \rangle e_i \right\|^2 = \left\| \sum_{i=k+1}^{+\infty} \langle \psi, e_i \rangle e_i \right\|^2 = \sum_{i=k+1}^{+\infty} \langle \psi, e_i \rangle^2
\]

on the other hand

\[
\|\nabla \psi\|^2 = \sum_{i=1}^{+\infty} \lambda_i \langle \psi, e_i \rangle^2 \geq \sum_{i=k+1}^{+\infty} \lambda_i \langle \psi, e_i \rangle^2 \geq \lambda_{k+1} \sum_{i=k+1}^{+\infty} \langle \psi, e_i \rangle^2
\]
Theorem 2 There is no \( k \geq 1 \) and no constant \( 0 \leq \alpha < 1 \) and no ordered basis \( \{ \beta_i \} \), such that

\[
\left\| \psi - \sum_{i=1}^{k} \langle \psi, \beta_i \rangle \beta_i \right\|^2 \leq \frac{\alpha \| \nabla \psi \|^2}{\lambda_{k+1}} \quad \forall \psi \in \{ \psi_\omega \}. \tag{1}
\]

**Proof.** Poincaré’s trick [3]. Assume there is, and let
\[
\psi = c_1 e_1 + c_2 e_2 + \cdots + c_k e_k + c_{k+1} e_{k+1}.
\]
The under-determined linear system

\[
\langle \psi, \beta_i \rangle = 0 \quad \forall i = 1, \cdots, k,
\]
of \( k \) equations with \( k+1 \) unknowns admits a non-trivial solution for \( \{ c_i \} \). Thus,

\[
\left\| \psi - \sum_{i=1}^{k} \langle \psi, \beta_i \rangle \beta_i \right\|^2 = \| \psi \|^2 = \sum_{i=1}^{k+1} c_i^2.
\]

Inserting \( \psi \) into (1) yields,

\[
\lambda_{k+1} \sum_{i=1}^{k+1} c_i^2 \leq \alpha \sum_{i=1}^{k+1} \lambda_i c_i^2 \leq \alpha \lambda_{k+1} \sum_{i=1}^{k+1} c_i^2.
\]

where we used the fact that \( \| \nabla \psi \|^2 = \sum_{i=1}^{k+1} \lambda_i c_i^2 \).

Therefore, \( \sum_{i=1}^{k+1} c_i^2 = 0 \Rightarrow c_i = 0 \) for all \( i \). A contradiction. \( \square \)
Theorem 3 Let the orthonormal basis \( \{ \beta_i \} \) satisfy for all \( k \geq 1 \),

\[
\left\| \psi - \sum_{i=1}^{k} \langle \psi, \beta_i \rangle \beta_i \right\|_2^2 \leq \frac{\| \nabla \psi \|^2}{\lambda_{k+1}} \quad \forall \psi \in \{ \psi_\omega \}. \tag{2}
\]

Then \( \beta_i \equiv e_i \) with corresponding eigenvalues \( \lambda_i \).

Proof.

Lemma 3.1 The basis signals for which (2) holds, satisfy

\[ \langle \beta_j, e_l \rangle = 0 \text{ for } 1 \leq j < l. \]

Proof. (HW)

Complete the uniqueness proof for the case of a simple spectrum

\[ 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots. \]

Use Poincaré’s “magic trick”, follow the steps in [3] while simplifying for the case of a simple spectrum.
References


Chapter Functional Maps

Chapter Functional Maps
Let \( f(x) : [0, 1] \rightarrow \mathbb{R} \) be a smooth function. Let \( \{ \phi_i(x) \}_{i=1}^{\infty} \) be an orthonormal basis defined on \( I = [0, 1] \), such that

\[
\langle \phi_i(x), \phi_j(x) \rangle_I \equiv \int_0^1 \phi_i(x) \phi_j(x) \, dx = \delta_{ij},
\]

where \( \delta_{ii} = 1 \) for all \( i \), and \( \delta_{ij} = 0 \) for \( i \neq j \).

We can represent \( f(x) \) in the basis \( \{ \phi_i(x) \} \), so that,

\[
f(x) = \sum_{i=1}^{\infty} \langle f(x), \phi_i(x) \rangle_I \phi_i(x)
\]

\[
= \sum_{i=1}^{\infty} \alpha_i \phi_i(x)
\]

\[
\approx \sum_{i=1}^{k} \alpha_i \phi_i(x).
\]
Define a continuous one-to-one mapping between the interval $I = [0, 1]$ and the interval $\tilde{I} = [a, b]$ to be $T(x) : I \rightarrow \tilde{I}$.
We have $\tilde{x} = T(x)$ and $x = T^{-1}(\tilde{x})$. The function $\tilde{f}(\tilde{x}) : \tilde{I} \rightarrow \mathbb{R}$ can be constructed by using $T$ to translate the coordinates of $f(x)$, namely, $\tilde{f}(\tilde{x}) = f(T^{-1}(\tilde{x}))$.
Now, let $\{\tilde{\psi}_i\}_{i=1}^\infty$ be an orthonormal basis defined on $\tilde{I}$. We can express $\tilde{f}(\tilde{x}) = \sum_{i=1}^\infty \tilde{\beta}_i \tilde{\psi}_i(\tilde{x})$. It also holds that for each basis function $\phi_i(x)$ its mapping to $\tilde{I}$ is given by $\tilde{\phi}_i(\tilde{x}) = \phi_i(T^{-1}(\tilde{x}))$.
We readily have that

$$\tilde{\phi}_i(\tilde{x}) = \sum_{j=0}^\infty \langle \tilde{\phi}_i, \tilde{\psi}_j \rangle_{\tilde{I}} \tilde{\psi}_j(\tilde{x})$$

$$= \sum_{j=0}^\infty \tilde{\gamma}_{ij} \tilde{\psi}_j(\tilde{x}), \quad (5)$$

where we define $\tilde{\gamma}_{ij} \equiv \langle \phi_i(T^{-1}(\tilde{x})), \tilde{\psi}_j \rangle_{\tilde{I}}$. 
Finally, note that we could write $\tilde{f}(\tilde{x})$ as a function of $\tilde{\psi}(\tilde{x})$ and $\phi(x)$. Specifically,

$$
\tilde{f}(\tilde{x}) = \sum_{i=0}^{\infty} \langle f(x), \phi_i(x) \rangle I_\phi_i(T^{-1}(\tilde{x})) \\
= \sum_{i=0}^{\infty} \alpha_i \tilde{\phi}_i(\tilde{x}) \\
= \sum_{i=0}^{\infty} \alpha_i \sum_{j=0}^{\infty} \tilde{\gamma}_{ij} \tilde{\psi}_j(\tilde{x}) \\
= \sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} \alpha_i \tilde{\gamma}_{ij} \right) \tilde{\beta}_j \tilde{\psi}_j(\tilde{x}) \\
= \sum_{j=0}^{\infty} \tilde{\beta}_j \tilde{\psi}_j(\tilde{x}).
$$

(6)
We see that in order to *translate* the $\alpha$ coefficients into $\tilde{\beta}$’s, we have to integrate them using the inner product between the two bases, namely, $\tilde{\gamma}_{ij} = \langle \tilde{\phi}_i(\tilde{x}), \tilde{\psi}_j(\tilde{x}) \rangle_{\tilde{I}}$. But, note that the translated coefficients $\tilde{\gamma}_{ij}$ do not depend on the function we would like to express. Thus, the knowledge of $T$ makes the relation between the decomposition of corresponding functions on two different domains a linear operation. Again, the knowledge of $T$ allows us to define the inner product between the eigenfunctions on one domain and eigenfunctions defined on another domain. This inner product is the way to translate decomposition coefficients from one domain to another.
The above simple observation required the mapping $T$ in order to find the relation between two domains $I$ and $\tilde{I}$. The question is what can be done if $T$ is unknown, but, say $\tilde{\gamma}_{ij} \approx \delta_{ij}$. This is the case, up to sign ambiguities, in isometric asymmetric surfaces when using the eigenfunctions of the corresponding Laplace-Beltrami operators. So, given corresponding eigenfunctions, all that is left is find corresponding points. This can be accomplished, for example, by translating the decomposition of a Gaussian from $I$ to $\tilde{I}$ and then setting the corresponding maxima as corresponding points.
The situation is less favorable when $\tilde{\gamma}_{ij} \neq \delta_{ij}$ and there is a need to find the correspondence and the coefficients simultaneously. In that case, constraints of probable matches, and penalizing for $|\tilde{\gamma}_{ij} - \delta_{ij}|$, could lead to the desired matching function $T$. Given the feature function

$$g(x) \approx \sum_{i=1}^{K} g_i \phi_i(x)$$

on $I$, and a corresponding feature function

$$\tilde{g}(\tilde{x}) \approx \sum_{i=1}^{K} \tilde{g}_i \tilde{\psi}_i(\tilde{x})$$

on $\tilde{I}$. We could assume that

$$(\tilde{g}_i)_{K \times 1} \approx (\tilde{\gamma}_{ij})_{K \times K}(g_i)_{K \times 1}.$$ 

Given more than such $K$ corresponding feature functions we could estimate the functional map $(\tilde{\gamma}_{ij})_{K \times K}$ in a least squares fashion and thereby solve for the mapping $T$. 

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Reference