Equi-Affine Geometry of Surfaces

Given a surface \( S : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3 \) given in a parametric form \( S(u, v) = (x(u, v), y(u, v), z(u, v)) \) and a parametric curve \( C(w) \in S \). The parametrization \( w \) would be equi-affine invariant if we restrict to one the volume of the parallelogram prism \( \det (C_{ww}, S_u, S_v) \) for any \( u, v \) parametrization of \( S \).

We could write

\[
1 = \det (C_{ww}, S_u, S_v) = \det (S_{ww}, S_u, S_v) \\
= \det \left( S_{uu} \left( \frac{du}{dw} \right)^2 + 2S_{uv} \frac{du}{dw} \frac{dv}{dw} + S_{vv} \left( \frac{dv}{dw} \right)^2 + S_u \frac{d^2 u}{dw^2} + S_v \frac{d^2 v}{dw^2}, S_u, S_v \right) \\
= \left( \frac{1}{dw} \right)^2 \det \left( S_{uu} du^2 + 2S_{uv} dudv + S_{vv} dv^2, S_u, S_v \right) \\
= \frac{1}{(dw)^2} \left( r_{11} du^2 + 2r_{12} dudv + r_{22} dv^2 \right),
\]

for the pre-metric \( r_{ij} \equiv \det (S_{ij}, S_u, S_v) \). For the metric to be re-parameterization invariant define \( q_{ij} = |r|^{-1/4} r_{ij} \).

Prove at home (recall the \( \kappa^{1/3} \)).
By Brioschi formula the Gaussian curvature $\mathcal{K}$ can be computed from the metric and its derivatives. Thereby, a full affine pseudo-metric $h_{ij} = |\mathcal{K}_q|q_{ij}$, is equi-affine invariant by construction and scale invariant as proven in [1]. The surface laplacian operator in both cases involves up to 3rd order derivative,

$$\Delta_h = \frac{1}{\sqrt{h}} \partial_i \sqrt{h} h^{ij} \partial_j.$$
Heat equation, kernel, and signature

The heat equation on the manifold $S$ is given by

$$\left( \frac{\partial}{\partial t} - \Delta_g \right) u(s, t) = 0.$$ 

The eigendecomposition of the LBO yields,

$$-\Delta_g \phi_i(s) = \lambda_i \phi_i(s),$$

by which the heat kernel is given as

$$K(s, s'; t) = \sum_{i=1}^{+\infty} e^{-\lambda_i t} \phi_i(s) \phi_i(s').$$

It solves the heat equation $\left( \frac{\partial}{\partial t} - \Delta_g(s) \right) K(s, s'; t) = 0.$
Discrete Laplacian and HKS

The heat kernel signature (HKS) is

\[ h(s, t) \equiv K(s, s; t) = \sum_{i=1}^{+\infty} e^{-\lambda_i t} \phi_i^2(s) = \frac{1}{4\pi} \left( t^{-1} + \frac{\mathcal{K}(s)}{3} + O(t) \right), \]

where \( \mathcal{K} \) is the Gaussian curvature.

The discrete LBO can be written as \( L = A^{-1} W = \Phi \Lambda \Phi^T A \) where \( A_{ii} \) is the area element about vertex \( i \) of the triangulated surface \( S \) and \( \Lambda_{ii} = \lambda_i \), in which case the discrete kernel can be written as

\[ K(t) = \Phi \exp(-t\Lambda) \Phi^T, \]

and the spatially discrete HKS is given by \( K_{ii}(t) \).
Diffusion distance

\[ d_{DD}^2(s, s'; t) \equiv \int_S (K(s, \bar{s}; t) - K(s', \bar{s}; t))^2 \, da(\bar{s}) \]

\[ = \int_S \left( \sum_{i=1}^{+\infty} e^{-\lambda_i t} \phi_i(s) \phi_i(\bar{s}) - \sum_{j=1}^{+\infty} e^{-\lambda_j t} \phi_j(s') \phi_j(\bar{s}) \right)^2 \, da(\bar{s}) \]

\[ = \int_S \left( \sum_{i=1}^{+\infty} e^{-\lambda_i t} \phi_i(s) \phi_i(\bar{s}) \right)^2 + \left( \sum_{j=1}^{+\infty} e^{-\lambda_j t} \phi_j(s') \phi_j(\bar{s}) \right)^2 - 2 \left( \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} e^{-\lambda_i t} \phi_i(s') \phi_i(\bar{s}) e^{-\lambda_j t} \phi_j(s') \phi_j(\bar{s}) \right) \, da(\bar{s}) \]

\[ = \sum_{i=1}^{+\infty} \left( e^{-2\lambda_i t} \phi_i^2(s) + e^{-2\lambda_i t} \phi_i^2(s') - 2 e^{-2\lambda_i t} \phi_i(s) \phi_i(s') \right) \]

\[ = \sum_{i=1}^{+\infty} e^{-2\lambda_i t} \left( \phi_i(s) - \phi_i(s') \right)^2 \]

where we used the fact that \( \langle \phi_i, \phi_j \rangle_g \equiv \int_S \phi_i(s) \phi_j(s) \, da(s) = \delta_{ij} \).
Diffusion distances in action

Given the regular metric $g_{ij}$, the scale invariant $\tilde{g}_{ij} = |K|g_{ij}$, the equi-affine metric $q_{ij}$ or the full affine $h_{ij} = |K_q|q_{ij}$, we could approximate the diffusion distances between surface points, plug it into the Gromov-Wasserstein or Gromov-Hausdorff (MDS, GMDS, SGMDS,..) frameworks, stabilize the numerical construction and make it robust to topological noise.
The scale invariant metric can be used to define a scale invariant LBO from which a scale invariant spectral geometry could be constructed. Such invariant metric by which surfaces are treated as conformal pseudo-Riemannian manifolds defines a laplacian that involves up to second order derivatives. This stability property of conformal geometry happens only for two dimensional manifolds. For triangulated surfaces, the LBO can be approximated by an equation that involves only the angles of the triangles. This construction allows stable approximation of quantities like its spectral decomposition for shape analysis.
In its continuous setting the LBO is defined as

$$\Delta_g \equiv \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j,$$

where by Einstein’s summation convention \((g_{ij})\) is the metric matrix and \(g \equiv \text{det}(g_{ij})\). One index as subscript followed by the same index as superscript, or vice-versa, means summation over the relevant indices. For two dimensional manifolds we have

\[
(g^{ij}) \equiv (g_{ij})^{-1} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}^{-1} = \frac{1}{g} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}.
\]
Given the scale invariant pseudo-metric $\tilde{g}_{ij} = |\mathcal{K}|g_{ij}$, we have

$$\tilde{g} = \mathcal{K}^2 g,$$

and

$$ (\tilde{g}^{ij} ) = \left( \begin{array}{cc} |\mathcal{K}|g_{11} & |\mathcal{K}|g_{12} \\ |\mathcal{K}|g_{12} & |\mathcal{K}|g_{22} \end{array} \right)^{-1}$$

$$= \frac{1}{\mathcal{K}^2 g} \left( \begin{array}{cc} |\mathcal{K}|g_{22} & -|\mathcal{K}|g_{12} \\ -|\mathcal{K}|g_{12} & |\mathcal{K}|g_{11} \end{array} \right)$$

$$= \frac{1}{|\mathcal{K}| g} \left( \begin{array}{cc} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{array} \right)$$

$$= |\mathcal{K}|^{-1}(g^{ij}).$$
Thus, for 2D manifolds (surfaces),

\[ \Delta_{\tilde{g}} \equiv \frac{1}{\sqrt{\tilde{g}}} \partial_i \sqrt{\tilde{g}} \tilde{g}^{ij} \partial_j \]

\[ = \frac{1}{\sqrt{\mathcal{K}^2 g}} \partial_i \sqrt{\mathcal{K}^2 g} |\mathcal{K}|^{-1} g^{ij} \partial_j \]

\[ = |\mathcal{K}|^{-1} \frac{1}{\sqrt{g}} \partial_i |\mathcal{K}| \sqrt{g} |\mathcal{K}|^{-1} g^{ij} \partial_j \]

\[ = |\mathcal{K}|^{-1} \Delta_g. \]

The resulting expression involves only second order derivatives required for the definition of \( \Delta_{\tilde{g}} \) and the Gaussian curvature. Although we consider a richer group of transformations the order of derivatives required for computing the laplacian remains the same.
Measures used to define $\Delta_g$ and $\mathcal{K}$.

Given a triangulated surfaces with $N$ vertices $\{v_1, \ldots, v_i, \ldots, v_N\}$, the one-ring neighborhood $\mathcal{N}_i$ is defined by the vertices connected by a single edge to $v_i$. The angular deficiency about $v_i$ is

$$Q_i \equiv \left| 2\pi - \sum_{v_j \in \mathcal{N}_i} \theta_j^i \right|,$$

where $\theta_j^i$ is the angle of the triangle with the edge $(v_i, v_j)$ for $v_j \in \mathcal{N}_i$. 


Angles are defined by the edge of a triangle visited in a counter-clockwise fashion about $v_i$. $|\kappa_i|$ at $v_i$ can be approximated by the Gauss-Bonet approximation

$$\kappa_i = \frac{Q_i}{A_i},$$

where, $A_i$ is the area about $v_i$, see [Pinkall et al. 1993]. As an operator in matrix setting, we can define $\kappa \equiv \text{diag}(\kappa_1, \ldots, \kappa_i, \ldots, \kappa_N)$ and similarly for $Q$ and $A$, by which

$$\kappa \equiv A^{-1}Q.$$
The cotangent weights matrix $W$ is

$$W_{ij} = \begin{cases} \sum_{v_j \in N_i} \omega_{ij} & \text{if } i = j \\ -\omega_{ij} & \text{if } i \neq j, v_j \in N_i \\ 0 & \text{otherwise,} \end{cases}$$

for $\omega_{ij} \equiv \cot \alpha_{ij} + \cot \beta_{ij}$, where $\alpha_{ij}$ and $\beta_{ij}$ are the angles opposite to the edge $\langle v_i, v_j \rangle$.

The LBO can be approximated by

$$L = A^{-1}W.$$

The scale invariant LBO is then approximated by

$$\tilde{L} = K^{-1}A^{-1}W$$

$$= (A^{-1}Q)^{-1}A^{-1}W$$

$$= (Q^{-1}A)A^{-1}W$$

$$= Q^{-1}AA^{-1}W$$

$$= Q^{-1}W.$$
The elements of $\tilde{L}$ read

$$\tilde{L}_{ij} = \begin{cases} 
\frac{\sum_{v_j \in \mathcal{N}_i} (\cot \alpha_{ij} + \cot \beta_{ij})}{|2\pi - \sum_{v_j \in \mathcal{N}_i} \theta^i_j|} & \text{if } i = j \\
-(\cot \alpha_{ij} + \cot \beta_{ij}) & \text{if } i \neq j, v_j \in \mathcal{N}_i \\
0 & \text{otherwise.}
\end{cases}$$

Computing the decomposition of this (possibly singular) operator, we resort to the generalized eigenvalue problem

$$W\tilde{\phi}_i = \lambda_i Q\tilde{\phi}_i.$$ 

This operator involves only angles of the given triangulated mesh.
Lipschitz embedding within $d_{GH}$, i.e.

$$
| \log d_S(s_i, s_j) - \log d_Q(q_i, q_j) | = \left| \log \frac{d_S(s_i, s_j)}{d_Q(q_i, q_j)} \right|
$$

- Topology constraint isometric embedding
- Beltrami flow (bilateral).
- Exaggeration for better recognition
Further reading


