Tutorial 7

Curve and surface evolution

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Evolution of planar curves

Consider a family $C(p, t) : (a, b) \times [0, T) \mapsto \mathbb{R}^2$ of parametric planar closed curves evolving in time according to the PDE

$$C_t(p, t) = V(p, t),$$

where $C(p, 0) = C_0(p)$ is some initial curve and

$$V(p, t) : (a, b) \times [0, T) \mapsto \mathbb{R}^2$$

is some velocity field.
The evolution equation can be rewritten by decomposing $V$ in the local frame $N, T$:

$$C_t(p, t) = \langle V, N \rangle N + \langle V, T \rangle T = V_N N + V_T T.$$ 

Every evolution equation can be written using the normal component only

$$C_t(p, t) = \langle V, N \rangle N$$

by an appropriate reparameterization.

Sign $C_t(p) = \pm \langle V, N \rangle N$: inward or outward flow direction.
Lemma [Epstein-Gage]: The family of curves $C(p, t)$ that solve the evolution rule $C_t = V_N N + V_T T$, where $V_N$ does not depend on the parameterization, can be converted into the solution of $C_t = V_N N$.

Proof:
Given $C(p, t) : S^1 \times [0, T) \rightarrow R^2$ as our family of curves, let $p = p(\omega, \tau)$ and $t = \tau$ with $\partial p / \partial \omega > 0$ be a reparametrization.

By the chain rule:
$$\frac{\partial}{\partial \tau} C(\omega, \tau) = \frac{\partial}{\partial \tau} C(p(\omega, \tau), t(\omega, \tau)) = C_p p_{\tau} + \partial_t C t_{\tau} = C_p p_{\tau} + \partial_t C$$
Lemma [Epstein-Gage] (cont.)

For the arclength parametrization $s$ we have

$$C_p = C_s s_p = Ts_p,$$

so

$$\frac{\partial}{\partial \tau} C = C_p p_{\tau} + \partial_t C = Ts_p p_{\tau} + V_T T + V_N N = (s_p p_{\tau} + V_T) T + V_N N$$

For each fixed $\omega$ we solve the O.D.E. $s_p p_{\tau} + V_T = 0$, and, since $t = \tau$, we get $C_t = V_N N$. 
Examples of evolution equations

**Constant flow:** $C_t = \mathcal{N}$.
Creates offsets of the initial curve. Shocks may appear.

**Curvature flow (geometric heat equation):** $C_t = \kappa \mathcal{N} = C_{ss}$.
Evolves any planar curve into a circular point in finite time [Gage-Hamilton, Grayson].

**Equi-affine invariant heat equation:** $C_t = \kappa \frac{1}{3} \mathcal{N}$.
Constant flow

Figure: Evolution of a curve under constant flow $C_t = \mathcal{N}$. 
Curvature flow

Figure: Evolution of a curve under curvature flow $C_t = \kappa \mathbf{N}$. 
Consider a family $S(u, v, t) : U \subset \mathbb{R}^2 \times [0, T) \rightarrow \mathbb{R}^3$ of parametric surfaces evolving in time according to the PDE
\[ S_t = F, \]
where $F$ is some smooth vector field.

As in the case of curves, the evolution equation can be rewritten as $S_t = \langle F, N \rangle N$, while the tangential part affects only the internal parameterization of $S$ and does not influence the shape.

Mean curvature flow: $S_t = HN$. 
Mean curvature flow of $S(u, v) = (u, v, z(u, v))$

Given $S$ as a graph of function $(u, v, z(u, v))$, we can rewrite the mean curvature flow as

$$S_t = \frac{H}{\langle N, Z \rangle} Z,$$

where $Z$ is some smooth vector field, e.g. $Z \equiv (0, 0, 1)$. In this case,

$$\frac{Z}{\langle N, Z \rangle} = \sqrt{1 + z_u^2 + z_v^2}(0, 0, 1) = \sqrt{g}(0, 0, 1),$$

i.e. it is enough to evolve only $z$:

$$z_t = H\sqrt{g}.$$
Exercise: Mean curvature flow

Derive the mean curvature flow of a function graph from the minimization of $A(S) = \int da$.

Solution
Let $S$ be a function graph given by $(u, v, z(u, v))$. The area element is given by $\sqrt{g} \triangleq \sqrt{EG - F^2}$, or in other words, $da = \sqrt{g}dudv = \sqrt{1 + z^2_u + z^2_v}dudv$.

$$
(G) = \begin{pmatrix}
1 + z^2_u & z_u z_v \\
z_u z_v & 1 + z^2_v
\end{pmatrix}
$$

$$
det(G) = EG - F^2 = 1 + z^2_u + z^2_v + z^2_u z^2_v - (z_u z_v)^2 = 1 + z^2_u + z^2_v
$$
Exercise: Mean curvature flow

Then,

\[ A(z) = \int \sqrt{g} \, dudv = \int \sqrt{1 + z_u^2 + z_v^2} \, dudv. \]

is the surface area.

The EL equation is given by (HW1):

\[ \frac{\partial}{\partial z} \sqrt{g} - \frac{\partial}{\partial u} \frac{\partial \sqrt{g}}{\partial z_u} - \frac{\partial}{\partial v} \frac{\partial \sqrt{g}}{\partial z_v} = - \frac{\partial}{\partial u} \frac{\partial \sqrt{g}}{\partial z_u} - \frac{\partial}{\partial v} \frac{\partial \sqrt{g}}{\partial z_v} = 0 \]
Exercise: Mean curvature flow (cont.)

which leads to

\[- \frac{\partial}{\partial u} \frac{z_u}{\sqrt{g}} - \frac{\partial}{\partial v} \frac{z_v}{\sqrt{g}} = \frac{z_{uu} \sqrt{g} - z_u \frac{\partial}{\partial u} \sqrt{g}}{g^{3/2}} - \frac{z_{vv} \sqrt{g} - z_v \frac{\partial}{\partial v} \sqrt{g}}{g^{3/2}} = 0\]

substituting \( \frac{\partial}{\partial u} \sqrt{g} = \frac{z_u z_{uu} + z_v z_{vu}}{\sqrt{g}} \) and \( \frac{\partial}{\partial v} \sqrt{g} = \frac{z_v z_{vv} + z_u z_{uv}}{\sqrt{g}} \), we obtain

\[- \frac{z_{uu}(1+z_v^2) - 2z_u z_v z_{uv} + z_{vv}(1+z_u^2)}{g^2} = \frac{z_{uu}(1+z_v^2) - 2z_u z_v z_{uv} + z_{vv}(1+z_u^2)}{g^{3/2}} \sqrt{g} = -H \sqrt{g} = 0.\]

Formulating the flow as a gradient descent \( z_t = -\frac{\delta A}{\delta z} \), we have the mean curvature flow \( z_t = H \sqrt{g} \).
Images as surfaces

A grayscale digital image can be though of as a surface represented as a function graph. The parameterization coordinates \((u, v)\) are identified with the pixel indices and \(z(u, v)\) represents the intensity level at each pixel.

Surface evolution results in a nonlinear PDE-based filtering of the image.
Figure: Noisy image
Figure: Denoising. Left: mean curvature flow. Right: Beltrami flow.

In grayscale - the Beltrami flow strengthens the edge preservation of the flow.
Curves as level sets

Given a graph of a function \((u, v, z(u, v))\) we can think of a curve \(C\) defined by (w.l.o.g.) the zero-set of \(z\), i.e. the trace of \(C\) is given by

\[
C = \{(u, v) : z(u, v) = 0\}.
\]

We will denote \(C = z^{-1}\). In images, level sets are *isophotes* (equal intensity curves).

Let \(C\) be a level set curve of \(z\) in arclength parameterization. It satisfies

\[
\langle \nabla z, C_s \rangle = 0,
\]

i.e. \(\nabla z = (z_u, z_v)\) is orthogonal to \(T = C_s\).
Figure: Level sets representation of a curve.
Evolution of level sets

Now assume we want to evolve the level set curve $C$ according to $C_t = V_N N$.

The corresponding evolution equation for the surface is

$$z_t = -V_N \|\nabla z\| = -V_N \sqrt{z_u^2 + z_v^2}.$$ 

**Proof:** Along the evolving curve $z$ is constant. Thus,

$$\frac{d}{dt} z(u(t), v(t), t) = 0 \Rightarrow z_t + z_u u_t + z_v v_t = 0$$

$$z_t = -\langle \nabla z, C_t \rangle = -\langle \nabla z, V_N N \rangle = -\langle \nabla z, V_N \frac{\nabla z}{\|\nabla z\|} \rangle = -V_N \|\nabla z\|$$
Exercise: curvature of $C = z^{-1}$

Show that the curvature of the planar curve $C = z^{-1}$ is given by

$$\kappa = -\frac{z_{uu}z_v^2 - 2z_u z_v z_{uv} + z_{vv} z_u^2}{(z_u^2 + z_v^2)^{3/2}}$$

**Solution:** Let $s$ be the arclength parameterization of $C$. Along the curve $C$ the function $z$, doesn't change it’s value, therefore $z_s = 0 = z_u u_s + z_v v_s$.

\[
\frac{\partial^2 z}{\partial s^2} = 0 = z_{uu} u_s^2 + 2z_{uv} u_s v_s + z_{vv} v_s^2 + \langle \nabla z, C_{ss} \rangle.
\]

Recall that $N = \nabla z/\|\nabla z\|$, $C_s = (u_s, v_s) = (N_2, -N_1) = ((\nabla z)_2, -(\nabla z)_1)/\|\nabla z\|$ and $C_{ss} = \kappa N$.

Therefore:

\[
0 = \frac{z_{uu}z_v^2 - 2z_u z_v z_{uv} + z_{vv} z_u^2}{\|\nabla z\|^2} + \|\nabla z\| \kappa \rightarrow
\]

$$\kappa = -\frac{z_{uu}z_v^2 - 2z_u z_v z_{uv} + z_{vv} z_u^2}{\|\nabla z\|^3}.$$
Exercise: Flows of curves and surfaces

Let $S$ be a graph of a function $(u, v, \alpha z(u, v))$. For which value of $\alpha$ evolving $S$ under the mean curvature flow is equivalent to evolving its level sets according to the curvature flow?

Solution

Let $S$ be a function graph given by $(u, v, \alpha z(u, v))$, and let the curve $C$ be (w.l.o.g.) the zero-set of $z$. We assume that $C$ evolves according to the curvature flow $C_t = \kappa \mathbf{N}$. The corresponding evolution equation of the surface $S$ is $z_t = -\kappa \| \nabla z \|.$

We will use the fact that the curvature of a level set curve of $z$ is given by

$$
\kappa = -\frac{z_{uu}z_v^2 - 2z_u z_v z_{uv} + z_{vv}z_u^2}{(z_u^2 + z_v^2)^{3/2}}
$$

(note: $\kappa$ is independent of $\alpha$).
Now consider the mean curvature of $S$ (HW3):

$$H = \frac{\alpha z_{uu}(1 + \alpha^2 z_V^2) - 2\alpha^3 z_u z_v z_{uv} + \alpha z_{vv}(1 + \alpha^2 z_{u}^2)}{(1 + \alpha^2 z_{u}^2 + \alpha^2 z_{v}^2)^{3/2}}$$

When $\alpha \gg 1$ (asymptotically, $\alpha \to \infty$), we have

$$H \approx \frac{z_{uu}z_v^2 - 2z_u z_v z_{uv} + z_{vv}z_u^2}{(z_u^2 + z_v^2)^{3/2}} = -\kappa$$

The level set evolution equation for $\alpha \gg 1$ becomes

$$z_t = -\kappa \| \nabla z \| \approx H \sqrt{\alpha^2 z_u^2 + \alpha^2 z_v^2} \approx H \sqrt{1 + \alpha^2 z_u^2 + \alpha^2 z_v^2} = H \sqrt{g},$$

i.e. the mean curvature flow.