Noise simulation

Exact

Most general: start with operator-sum

\[ E(p) = \sum_{\mu} E_{\mu} E_{\mu}^* \]

\[ E_{\mu} = \langle \phi_{\mu} | \phi_{\mu} \rangle \]

\[ \sum_{\mu} E_{\mu}^* E_{\mu} = 1 \]

\[ \rho \rightarrow E_{\mu} \rho E_{\mu}^* \]

\[ \text{tr}(E_{\mu} \rho E_{\mu}^*) \]

Exact simulation of quantum states:

\[ |\phi\rangle \rightarrow U |\phi\rangle = |\psi\rangle \]

\[ U : 2^n \times 2^n \text{ matrix}, \]

\[ |\psi\rangle : 2^n \text{ state vector} \]

For Exact (naive) noise simulation:

\[ \rho \rightarrow E(p) = \sum_{\mu} E_{\mu} \rho E_{\mu}^* = \rho' \]

\[ \text{size } \rho \sim |\phi\rangle \langle \phi | \sim 2^n \]

Noise vs. exact naive simulation of noise: \[ 2^n \]
initial $\psi$:

$$14 \otimes \psi = \sum_{\alpha} \alpha_1 10 \otimes \psi_1 + \sum_{\beta} \beta_1 11 \otimes \psi_1$$

(c.e. $\alpha_1 10 01 10 02 + \beta_1 11 02 10 02 = \alpha_1 10 \otimes \psi_1 + \beta_1 11 \otimes \psi_1$)

$\phi = 14 \otimes \psi_1 =$

$$= \left( \sum_{\alpha} \psi_1 \right) \left( \sum_{\alpha} \psi_1 \right)$$

$$= \sum_{\alpha} \psi_1 \otimes \psi_1$$

Partial trace on $\otimes_2$: $P = \text{Tr}_2 (\psi) + \text{Tr}_2 (14 \otimes \psi_1)$
\[
\begin{aligned}
\rho^0 &= \sum_{y=1}^{2^{n-1}} \alpha_j \rho^{0} \langle \Psi_i | \Psi_j \rangle + \sum_{i,j=1}^{2^n-1} \langle \Psi_i | \Psi_j \rangle \rho^{*} \langle \Psi_i | \Psi_j \rangle + \sum_{i,j=1}^{2^n-1} \langle \Psi_i | \Psi_j \rangle \rho^{*} \langle \Psi_i | \Psi_j \rangle \\
&= \sum_{i=1}^{2^n-1} \left( \sum_{j=1}^{2^{n-1}} \alpha_j \right) \rho^{0} + \sum_{i=1}^{2^n-1} \langle \Psi_i | \Psi_i \rangle \rho^{*} \langle \Psi_i | \Psi_i \rangle \sum_{j=1}^{2^{n-1}} \alpha_j
\end{aligned}
\]
\[ p_n = \text{Tr} \left( E_n^+ E^+ \right) \]

\[
\sum_{m=1}^{m} E_{p_n} = \sum_{m=1}^{m} \text{Tr} \left( E_m^+ E_m^+ \right) = \text{Tr} \left( \sum_{m=1}^{m} E_m^+ E_m^+ \right) = \text{Tr} \left( \sum_{k=1}^{m} E_{k_n}^+ E_{k_n}^+ \right)
\]

Define:
\[ \rho_{\text{noise}} = \frac{1}{m} \sum_{k=1}^{m} \rho_k \]

\[ \rho_k \text{ are proper density matrices} \]

Positive:
for any \( \psi \):
\[
\langle \psi | \rho_k | \psi \rangle = \frac{1}{m} \langle \psi | \left( E_k \otimes I_{2n-2} \right) \rho \left( E_k^+ \otimes I_{2n-2} \right) | \psi \rangle =
\]

\[
\langle \psi | \rho_k | \psi \rangle \geq 0
\]

with \( \rho_k = \frac{1}{m} \left( E_k \otimes I_{2n-2} \right) \rho \left( E_k^+ \otimes I_{2n-2} \right) \]

\[ \text{Tr}(\rho_k) = 1 \text{ by (*) and properties of partial trace} \]

Choose \( r \) according to (*)

\[ \frac{1}{r} \rho \text{ on the state of first qubit.} \]
Approximate simulations

Clifford group, stabilizer formalism

Example: \( |\Psi \rangle = \frac{|111 \rangle + |000 \rangle}{\sqrt{2}} \)

\[X_1 X_2 |\Psi \rangle = \frac{|111 \rangle + 100 \rangle}{\sqrt{2}} = |\Psi \rangle\]

\[Z_1 Z_2 |\Psi \rangle = \frac{100 \rangle + (111 \rangle)}{\sqrt{2}} = |\Psi \rangle\]

\(|\Psi \rangle\) is stabilized by \(X_1 X_2, Z_1 Z_2\)

\(|\Psi \rangle\) is the only state stabilized by \(X_1 X_2, Z_1 Z_2\)

Pauli groups \(B_n\)

\[B_1 = \{1, i, iX, iY, iZ, iXZ, iYZ, iZX, 1, -1, i, -i, X, Y, Z, iX, iY, iZ, iXZ, iYZ, iZX\}\]

\[B_n = B \otimes B \otimes \cdots \otimes B\]

\(S \subseteq B_n\)

\(V_S\) is the set of \(n\)-qubit states which are fixed by every element of \(S\)

\(V_S\) is the vector space stabilized by \(S\)

\(S\) is the stabilizer of \(V_S\)
14^7, 14^2 \in V_5, s \in S

s(14^7 + 614_0) = a_514^7 + b_514_0 = a_14^7 + b_14_0

\Rightarrow V_5 \text{ a subspace of } H

V_5 \text{ is the intersection of all stabilizers of subsets.}

\text{Example: } n = 3, S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}

\{2, 3\} \text{ spanned by } 1000\overrightarrow{7}, 1001\overrightarrow{7}, 1100\overrightarrow{7}, 1111\overrightarrow{7}

\{2, 3\} \rightarrow 1000\overrightarrow{7}, 1100\overrightarrow{7}, 1011\overrightarrow{7}, 1111\overrightarrow{7}

S \rightarrow 1000\overrightarrow{7}, 1111\overrightarrow{7} \Rightarrow V_5 = a_1000\overrightarrow{7} + b_{1111\overrightarrow{7}}

description of group by generators: \( G = \langle 1, 2, \ldots, g \rangle \)

each element in \( G \) is products of the

\[ S = \langle 2, 3, 4, 5 \rangle \]

\[ \overrightarrow{2, 3} = (\overrightarrow{2, 3}) \overrightarrow{(2, 3)} \quad I = (\overrightarrow{2, 3}) \overrightarrow{(2, 3)} \]

compact representation: \( G \) has at most \( 16 \) generators

\( V_5 \) is stabilized already by the generators of \( S \)
V₃ + 105 giff:

1) the elements of S commute
2) -I # S

If -I ∈ S: 
- I stabilizes only 14₂ = 0

If M, N ∈ S do not commute then they anti-commute (Pauli)

\[ S₀ - NN = AN \]

\[ -14₂ = -NN₁₁₄₂ = MN₁₁₄₂ = 14₂ \quad \Rightarrow 14₂ = 0 \]

Want minimal (independent) set of generators

check matrix notation:

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Forbids:

\[
\begin{align*}
00 & \rightarrow \text{I} \\
01 & \rightarrow \text{Z} \\
10 & \rightarrow \text{X} \\
11 & \rightarrow \text{Y}
\end{align*}
\]
\[ s = \{ g, -g \}, \quad I + s \]

\[ s_i - s_e \text{ independent iff rows of } A \text{ are linearly independent} \]

From now on, \( s_i - s_e \) are independently connected, and it is

\[ \lambda = n - k \text{ iff } V_s \text{ is } 2^k \text{-dimensional} \]

**Dynamics (unitary gates)**

Let \( 147 \in V_s \) for some \( s = \{ g, -g \}, \)

\[ U147 = Ug147 = U_{s_i} V^*_{s_e} U147 \]

So \( U147 \) is stabilized by \( U_{s_i} V^*_{s_e} \)

\[ U V_3 \text{ is stabilized by } U S U^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U \]

\[ U V_3 = \begin{pmatrix} U_{s_i} & U_{s_e} \\ U_{s_i} & U_{s_e} \end{pmatrix} \]

**Example:** \( U = H \):

\[ H X H^* = 2 \quad H Y H^* = -Y \quad H Z H^* = X \]

\[ S = \{ 23 \}, \quad V_s = 10 \Rightarrow S' = \{ 2, 3 \}, \quad V_{s'} = 1 \Rightarrow \]
Examples:

**In qubits:**

\[ \text{Start } |i\rangle = |0^n \otimes \alpha \rangle \]

\[ S = \{ z_i, -z_i \} \]

Apply \( H \) to each of the qubits:

\[ S' = \{ x_i, -x_i \} \]

\[ +z_i z_i = +z_i \]

Compared to \( \sigma \) in the stabilizer formalism.

**Example 3:**

Entanglement

\[ U = CNOT \]

\[ U X_1 U^* = U (X \otimes I) U^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ U X_1 U^* = X_1 \]

\[ U z_1 U^* = z_1 \]

\[ U z_2 U^* = z_2 \]

From these all are calculable:

\[ U X_1 X_2 U^* = U X_1 U^* U X_2 U^* = X_1 X_2 X_2 = X_1 X_2 \]

\[ U X_2 U^* = i U X_2 Z_2 U^* = i U X_2 U^* U Z_2 U^* = i X_2 Z_2 \]

\[ U Z_1 U^* = Z_1 \]
<table>
<thead>
<tr>
<th>Operation</th>
<th>Input 1</th>
<th>Input 2</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>CNOT</td>
<td>$X_1$</td>
<td>$X_2$</td>
<td>$X_1X_2$</td>
</tr>
<tr>
<td>(control, target)</td>
<td>$X_2$</td>
<td>$X_2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Z_1$</td>
<td>$Z_1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Z_2$</td>
<td>$Z_1Z_2$</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
H & \quad X & \equiv Z \\
Z & \equiv X \\
(10) & \quad S & \equiv Z \\
(01) & \quad S & \equiv Z \\
\end{align*}
\]

Normalizer of $6_\text{m}$: all $U$ such that $U6_\text{m}U^\dagger = 6_\text{m}$

Any $U$ in normalizer can be composed from $H$, $S$, $\text{CNOT}$

$T$ not in $M(6_\text{m})$

$T^\dagger = \frac{x+y}{x}$
Clifford group: \( \mathcal{M}(G) \)
generated solely by \( \text{CNOT}, H, S \)

**Gottesman-Knill Theorem:**

A quantum circuit with the following elements:
1. qubits prepared in computational basis
2. gates from Clifford group
3. measurements in computational basis

Can be simulated classically in polynomial time

(best: \( O(n \log n) \))

**Applications:**
- purification
- quantum error correction
- simulation of noise
- random benchmark

*Cannot be used for simulation of actual circuits with quantum speedup*
simulation of noise:

start with any noise model

\[ \mathcal{E} \mathcal{E} \mathcal{E} \mathcal{E} \]

approx model \( \mathcal{E} \) as elements from \( \mathcal{M}(\mathcal{M}) \)

with parameters

optimize according to \( \mathcal{E} \) Trace distance

maintain proofs of honesty (or other promises)

random benchmarking:

Apply sequence of gates from \( \mathcal{M}(\mathcal{M}) \)

and compare to simulated fidelity

\[ \text{depth of seq} \]

apply different types of sequences.