Homework 2

Submission: July 5\textsuperscript{th} 2018

1. \textbf{The Embedding of Tree Metrics into} $\ell_1$ (20 Points)

Recall that a tree metric $(X, d_T)$ satisfies that there exists a tree $T = (X, E)$ equipped with weights $w : E \to \mathbb{R}_+$ such that if $P_{u,v}$ denotes the single path in $T$ between $u$ and $v$ then: $d_T(u,v) = \sum_{e \in P_{u,v}} w_e$. Prove that every tree metric $(X, d_T)$ can be isometrically embedded into $\ell_1^{n-1}$ (here $n = |X|$).

2. \textbf{Metric Decomposition into Bounded Diameter Clusters} (40 Points)

Given a metric space $(V, d)$ on $n$ points consider the following algorithm for decomposing $V$ into clusters of diameter at most $\Delta$:

(a) $\mathcal{C} \leftarrow \emptyset$.

(b) While $V \neq \emptyset$:

i. Pick an arbitrary $s \in V$.

ii. Choose $R \sim \exp(\lambda, \Delta)$ with $\lambda = \frac{2 \ln n}{\Delta}$ and $\alpha = \frac{\Delta}{2}$.

iii. $C \leftarrow \{ v \in V : d(s,v) \leq R \}$.

iv. $\mathcal{C} \leftarrow \mathcal{C} \cup \{ C \}$ and $V \leftarrow V \setminus C$.

(c) Return $\mathcal{C}$.

Here $\exp(\lambda, \alpha)$ denotes the exponential distribution with rate $\lambda$ truncated at $\alpha$, i.e., its density is:

$$f(t) = \begin{cases} \frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda \alpha}} & 0 \leq t \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

Prove that once the algorithm terminates, for every pair of points $u, v \in V$ the following holds:

$$\Pr [u \text{ and } v \text{ are in different clusters}] \leq O\left(\frac{\ln n}{\Delta} \cdot d(u,v)\right)$$

3. \textbf{Universal Traveling Salesperson Problem} (40 Points)

Given a metric space $(V, d)$ on $n$ points, the goal is to find a hamiltonian cycle $C$ that is universal in the following sense. For every $\emptyset \neq S \subseteq V$, let $C_S$ be the cycle $C$ induces on $S$, and $OPT_S$ the shortest hamiltonian cycle on $S$. Denote by $d(C_S)$ and $d(OPT_S)$ the lengths of $C_S$ and $OPT_S$ correspondingly. A hamiltonian cycle $C$ is $\alpha$-universal if for every $\emptyset \neq S \subseteq V$:

$$\frac{d(C_S)}{d(OPT_S)} \leq \alpha$$

Present an algorithm that finds a hamiltonian cycle $C$ achieving $\alpha = O(\log n)$. 

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