236610
Distributed Graph Algorithms

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Coloring a ring

• A ring is 3-colorable
  – A ring of even length is 2-colorable
  – 2-coloring a ring of even length requires $\Omega(n)$ rounds

• What about 3-coloring a ring?
  – We saw $(\Delta+1)$-coloring algorithms for general graphs
  – Can do much better: $O(\log^*n)$ rounds
Coloring a ring

• We find a $6$-coloring algorithm for the ring in $O(\log^* n)$ rounds

• Then use the color reduction algorithm from $c=6$ colors to $(\Delta+1)=3$ colors
  – An additional $O(1)$ rounds

• We assume the ring is oriented
  – Assumption can be removed
6-Coloring a ring

[Cole-Vishkin]

1. $\varphi(v) \leftarrow \text{ID}_v$

2. For $O(\log^*n)$ rounds

3. $i \leftarrow \min\{ k \mid \varphi(v)[k] \neq \varphi(\text{parent}(v))[k] \}$

4. $\varphi(v) \leftarrow <i, \varphi(v)[i]>$

5. send $\varphi(v)$ to all neighbors

6. return $\varphi(v)$
6-Coloring a ring

- **Illustration**

- **Claim 1**: At the end of each iteration, $\varphi$ is a valid coloring

- **Claim 2**: After $O(\log^* n)$ rounds, $\varphi$ is a 6-coloring
6-Coloring a ring

• **Claim 1**: At the end of each iteration, $\varphi$ is a valid coloring

• **Proof**: Let $v, w$ be neighbors, $w = parent(v)$. By induction, $\varphi(v) \neq \varphi(w)$ at the end of each iteration.

• **Base case**: IDs are unique.
6-Coloring a ring

• **Claim 1:** At the end of each iteration, \( \varphi \) is a valid coloring

• **Induction step:** Let \( i_v, i_w \) be the indices chosen in Line 3. If \( i_v \neq i_w \) then \( \varphi(v) \neq \varphi(w) \) after Line 4.

• Otherwise, let \( i = i_v = i_w \). Since at the beginning of the iteration \( \varphi \) is valid, \( \varphi(v)[i] \neq \varphi(w)[i] \), and hence \( \varphi(v) \neq \varphi(w) \) after Line 4.
6-Coloring a ring

• **Claim 2:** After $O(\log^* n)$ rounds, $\varphi$ is a 6-coloring.

• **Proof:** Each iteration that starts with a coloring of $b$ bits (a $2^b$-coloring) ends with a coloring of $\log b + 1$ bits (index and its value). After $O(\log^* n)$ iterations we get a coloring with 3 bits (an 8-coloring).

• But actually this is a 6-coloring: the first two bits represent an index, so 11 cannot occur.
3-coloring a ring

• Can be extended to \((\Delta + 1)\)-coloring:
  – In each iteration the new color is a sequence of \(\Delta\) pairs \(<\text{index}, \text{value}>\)
  – A \(b\)-bit coloring becomes a \(\Delta(\log b + 1)\)-bit coloring
  – \(O(\Delta)\)-bit coloring after \(O(\log^* n)\) rounds
  – Color reduction to \(\Delta + 1\) colors in a number of rounds that is exponential in \(\Delta\)
    • Constant number of rounds if \(\Delta\) is constant

• Can be extended to rooted oriented trees
Lower bound for $(\Delta+1)$-coloring

- **Theorem**: Any algorithm $A$ for 3-coloring a ring requires $\Omega(\log^* n)$ rounds

- **Definition**: A $(c,k)$-coloring function is a function $\varphi : X_k \rightarrow [c]$ where $X_k$ is the set of all vectors of $k$ different IDs, such that $\varphi(x_0,\ldots,x_{k-1}) \neq \varphi(x_1,\ldots,x_k)$ for every $x_0 < x_1 < \ldots < x_k$
Lower bound for \((\Delta+1)\)-coloring

- **Claim 1**: If \(A\) finishes in \(t\) rounds then \(A\) is a \((3,2t+1)\)-coloring function

- **Proof**: The output of a node \(v\) is determined by its \(t\)-neighborhood (lemma from first class). Hence, we can view \(A\) as a function

\[
A: X_{2t+1} \rightarrow \{0,1,2\}
\]

such that

\[
A(x_0,...,x_{2t}) \neq A(x_1,...,x_{2t+1})
\]

for every \(x_0,...,x_{2t+1}\) that are all different.

- In particular, this holds for every \(x_0 < x_1 < ... < x_{2t+1}\)
Lower bound for \((\Delta+1)\)-coloring

- **Claim 2**: If there is a \((c,1)\)-coloring function \(f\) then \(c \geq n\).

- **Proof**: By definition, \(f(x_0) \neq f(x_1)\) for every \(0 \leq x_0 < x_1 < n\), so \(f\) is 1:1, and hence \(c \geq n\).
Lower bound for \((\Delta+1)\)-coloring

- In other words: without communication, we must use \(n\) colors.

- **Main goal**: show that reducing the number of colors requires communication
Lower bound for \((\Delta+1)\)-coloring

- **Claim 3**: If there is a \((c,k)\)-coloring function \(f\) then there is a \((2^c, k-1)\)-coloring function \(g\).

- **Proof of Theorem**: By **Claim 1**, \(A\) is a \((3,2t+1)\)-coloring function. Applying **Claim 3** for \(2t\) times gives that there is a \((T,1)\)-coloring function, where \(T=2^2\ldots^2^3\) (\(t\) times). By **Claim 2** we have \(T\geq n\), and hence \(t\geq \Omega(\log^*n)\).
Lower bound for \((\Delta+1)\)-coloring

- **Claim 3**: If there is a \((c,k)\)-coloring function \(f\) then there is a \((2^c, k-1)\)-coloring function \(g\).

- **Proof**: Consider a bijection \(h:2^c \rightarrow 2^c\), mapping subsets of \([c]\) into \([2^c]\) (think characteristic vector).

- Define \(g: X_{k-1} \rightarrow 2^c\) by
  \[
g(x_0, \ldots, x_{k-2}) = h(\{f(x_0, \ldots, x_{k-1}) \mid x_{k-2} < x_{k-1} < n\})\]
Lower bound for $(\Delta+1)$-coloring

- Define $g: X_{k-1} \rightarrow [2^c]$ by
  $$g(x_0,\ldots,x_{k-2})=h( \{ f(x_0,\ldots,x_{k-1}) \mid x_{k-2}<x_{k-1}<n \} )$$

- That is, a vector of length $k-1$ is assigned by $g$ a color that is given by $h$ to the set of colors assigned by $f$ to all possible extensions of length $k$ of the vector
Lower bound for \((\Delta+1)\)-coloring

- We need to show that \(g(x_0,\ldots,x_{k-2}) \neq g(x_1,\ldots,x_{k-1})\) for all \(x_0 < x_1 < \ldots < x_{k-2} < x_{k-1} < n\).

- We have \(f(x_0,\ldots,x_{k-2},x_{k-1}) \neq f(x_1,\ldots,x_k)\) for all \(x_0 < x_1 < \ldots < x_{k-2} < x_{k-1} < x_k < n\) because \(f\) is a \((c,k)\)-coloring.

- So \(f(x_0,\ldots,x_{k-2},x_{k-1})\) which belongs to the set that defines \(g(x_0,\ldots,x_{k-2})\), does not belong to the set that defines \(g(x_1,\ldots,x_{k-1})\), giving \(g(x_0,\ldots,x_{k-2}) \neq g(x_1,\ldots,x_{k-1})\).
Lower bound for (Δ+1)-coloring

• **Theorem**: Any algorithm $A$ for 3-coloring a ring requires $\Omega(\log^* n)$ rounds

[Linial] [this version by Laurinharju-Suomela]

• Remarkably, we do not have stronger lower bounds for $(\Delta+1)$-coloring

• Next: additional local problems