Problem 1.
Let $C_1$ and $C_2$ be linear codes of the same lengths over a field $F = GF(p)$ and let $G_1$ and $G_2$ be generator matrices of $C_1$ and $C_2$, respectively. Define the following codes:

- $C_3 = C_1 \cup C_2$
- $C_4 = C_1 \cap C_2$
- $C_5 = C_1 + C_2 = \{ u + v : u \in C_1, v \in C_2 \}$

For $1 \leq i \leq 5$, denote by $k_i$ the dimension $\log |F| |C_i|$ and by $d_i$ the minimum distance of $C_i$. Assume that $k_1, k_2 > 0$.

1. Show that $C_3$ is linear if and only if $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.
2. Show that the codes $C_4$ and $C_5$ are linear.
3. Show that if $k_4 > 0$ then $d_4 \geq \max\{d_1, d_2\}$.
4. Show that $k_5 \leq k_1 + k_2$ and that the inequality is tight if and only if $k_4 = 0$.
5. Show that $d_5 \leq \min\{d_1, d_2\}$.

Problem 2.
Prove that for any linear $[n, k, d]$ code over a field $F$ of size $p$ it holds that

$$k \leq n - \log_p \left( \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{n}{i} (p-1)^i \right).$$

Problem 3.
1. Show that the number of full rank binary $m \times n$ matrices, where $m \leq n$, is $\prod_{i=0}^{m-1} (2^n - 2^i)$.
2. Calculate the probability that a uniformly random binary $m \times n$ matrix, where $m \leq n$, is of full rank.
Problem 4.

1. Write the capacity region of a four-write WOM codes and find the probabilities $p_1, p_2, p_3$ that maximize the sum-rate to achieve value $\log_2 5$.

2. Find the maximum sum-rate of fixed-rate four-write WOM codes.

Problem 5.

In this problem, we will prove that $\log_2 (t + 1)$ is an upper bound on the sum-rate of any $t$-write WOM code.

Assume $C$ is an $[n, t; M_1, M_2, \ldots, M_t]$ $t$-write WOM code. For any sequence of messages $(m_1, \ldots, m_t)$, where for $1 \leq i \leq t$, $m_i \in \{1, \ldots, M_i\}$, $A(m_1, \ldots, m_t)$ is an array of size $t \times n$ such that its $i$-th row, for $1 \leq i \leq t$, is the cell-state vector on the $i$-th write.

1. Write the matrix $A$ for the Rivest Shamir code if the first two information bits are 10 and the second two information bits are 11.

2. Show that for any two different sequences of messages $(m_1, \ldots, m_t) \neq (m'_1, \ldots, m'_t)$, $A(m_1, \ldots, m_t) \neq A(m'_1, \ldots, m'_t)$.

3. Let $A$ be a matrix of size $t \times n$ which is equal to $A(m_1, \ldots, m_t)$ for some sequence of messages $m_1, \ldots, m_t$. How many different options does every column of $A$ have? How many different such matrices $A$ can exist?

4. Conclude that the sum-rate of $C$ is at most $\log_2 (t + 1)$, that is,

$$\sum_{i=1}^{t} \log_2 M_i \leq \log_2 (t + 1).$$

Problem 6. In this problem, we will study how to build a computer program that searches for two-write WOM codes. Assume there are $n$ cells and $0 \leq k \leq n$ is a given parameter. We will construct a bipartite graph consisting of two groups of vectors: $S_L$ and $S_R$. On its left side, the set $S_L$ consists of all binary vectors of weight at most $k$ and on its right side, the set $S_R$ consists of all binary vectors of weight at least $k$. There is an edge between a vector $v_L \in S_L$ and a vector $v_R \in S_R$ if and only if $v_L \leq v_R$.

1. Draw the bipartite graph for $n = 4$ and $k = 2$.

A WOM code, according to this bipartite graph, will enable to write $M_1 = \sum_{i=0}^{k} \binom{n}{i}$ messages on the first write, which are encoded to all possible vectors of the first group $S_L$. On the second write, we want to design the WOM code such that $M_2$ messages can be written. Assume one assigns a mapping $F : S_R \rightarrow \{1, \ldots, M_2\}$.

2. Explain how it is possible to check whether the mapping $F$ will generate a WOM code that allows to write on the worst case $M_2$ messages on the second write.

3. Design an algorithm (the best you can think of) to find such a mapping $F$ and explain how the value $M_2$ for the number of messages on the second write is derived from this algorithm.