Flash memories consist of floating-gate cells that can typically store a single bit, two bits, or three bits. A group of cells constitute a page, which is the smallest write unit, and the pages are organized in blocks, which are the smallest erase unit. The cells are organized into blocks which usually contain 64-512 pages, where the size of a page ranges between 2KB and 32KB. Usually (unless rewriting codes are used) pages can be updated only if their accommodating block is first erased, write update requests are performed out-of-place. Thus, when a page is updated, its previous location is marked as invalid. In order to accommodate this write procedure, the amount of physical storage has to be larger than the available logical storage. The ratio between the size of additional storage and logical storage is called over-provisioning. Furthermore, whenever there are no available blocks to accommodate page write requests, garbage collection (GC) is invoked to clean, i.e. erase, blocks for additional page writes. However, when a block is chosen to be cleaned by GC, its valid pages must be read and rewritten to a clean block, thereby increasing the total number of pages written to the memory. The write amplification is the ratio between the number of physical page writes and the number of logical page writes. This ratio is greater than one due to the additional page writes during the process of GC.

In fact, flash memories impose the constraint in which the level of each cell can only increase, and can be decreased only if its entire block is first erased. Thus, a WOM-code can be applied in flash memories to enable additional writes without first having to erase the block.

Reducing the write amplification is crucial as it directly affects the memory performance and its lifetime. In general, there is a direct relation between over-provisioning and write amplification. Increasing over-provisioning reduces the write amplification. However, high over-provisioning means that a large area of the memory is not exploited to store information. Thus, understanding the connection between the two measures is very important for optimizing the design of flash memories.

This write procedure introduces the following concepts:

- **Flash Translation Layer (FTL):** The FTL is responsible for mapping logical locations to physical ones.

- **Over-provisioning (OP):** The ratio between the amounts of additional storage and logical storage. This overhead is necessary to accommodate out-of-place writes.

- **Garbage Collection (GC):** The process in charge of cleaning blocks in order to free more space for writing.

- **Write Amplification (WA):** The ratio between the number of physical page writes and the number of logical page write requests.
The following summarizes the setup, notations, and assumptions we use in this study.

1. Every block has $Z$ pages, each of size $s$ KB. There are $T$ physical pages and $U$ logical pages, where both $T$ and $U$ are a multiple of $Z$.

2. The over-provisioning is $\rho = (T - U)/U$ and $\alpha = U/T = 1/(\rho + 1)$ is the storage rate, which is the ratio between logical data and physical storage.

3. $WA = P/L$, where $L$ is the number of write requests of logical pages and $P$ is the number of resulting physical page writes. We also define $L/Z$ to be the number of logical block writes.

The main goal is to study the connection between the over-provisioning $\rho$ (or storage rate $\alpha$) and the number of block erasures. This connection depends upon the over-provisioning value, GC algorithm, and the probability distribution of the page write requests\(^1\). We assume that requests are uniformly distributed over the $U$ logical pages. We assume that greedy GC is optimal for uniform distribution, where greedy GC always chooses the block with the minimum number of valid pages for cleaning. We also assume that greedy garbage collection is invoked whenever there are no more clean blocks. That is, we don’t require a minimum fraction of available blocks since the analysis is very similar to the one without this requirement.

WOM codes allow to write the blocks multiple times before an erasure. Thus, WA is not the right figure of merit since it is possible to write more pages and yet erase less. Hence, we introduce a new measure that better characterizes this behavior.

**Definition 1.** The erasure factor $EF$ in a flash memory system is the ratio between the number of block erasures $E$ and the number of logical block writes $L/Z$. That is,

$$EF = \frac{E}{L/Z}.$$  

Note that if no rewriting code is used then $EF = WA$.

**Theorem 2.** The number of block erasures $E$ and the erasure factor $EF_G(\alpha)$ for greedy GC (without rewriting) are given by

$$E = \frac{P}{L} = \frac{L}{Z(1 - \alpha')}, \quad EF_G(\alpha) = \frac{1}{1 - \alpha'} = \frac{1}{1 + \alpha \cdot W\left(-\frac{1}{\alpha} e^{-\frac{1}{\alpha}}\right)},$$  

where $\alpha = \frac{\alpha' - 1}{\ln(\alpha')}$, or $\alpha' = -\alpha \cdot W\left(-\frac{1}{\alpha} e^{-1/\alpha}\right)$, and $W(x)$ is the Lambert $W$ function.

**Proof.** For $0 \leq i \leq Z$, let $N(i)$ be a random variable corresponding to the number of blocks with $i$ valid pages, so $\sum_{i=0}^{Z} N(i) = T/Z$. If we denote by $Y$ the expected number of valid pages when a block is erased, then for $0 \leq i \leq Y - 1$, $N(i) = 0$, and $N(Y)$ is relatively small enough. We assume that the system is in steady state and thus the expected value of $N(i)$ doesn’t change over time\(^2\). According to this assumption, we also get that for $Y + 1 \leq i \leq Z$,

$$iN(i) = C,$$

\(^1\)and also on $Z$ but we assume in the paper that $Z$ is large enough to avoid this dependency.

\(^2\)These properties model this process as a Markov chain and the number of blocks with a given number of valid pages is fixed for analysis purposes.
for some constant $C$, or $N(i) = (Y + 1)N(Y + 1)/i$. Therefore, we get

$$T/Z = \sum_{i=0}^{Z} N(i) = \sum_{i=Y+1}^{Z} N(i) = \sum_{i=Y+1}^{Z} (Y + 1)N(Y + 1)/i$$

$$= (Y + 1)N(Y + 1) \sum_{i=Y+1}^{Z} \frac{1}{i}$$

$$\approx (Y + 1)N(Y + 1)(\ln(Z) - \ln(Y))$$

$$= (Y + 1)N(Y + 1)\ln(Z/Y).$$

We also have that

$$U = \sum_{i=0}^{Z} iN(i) = \sum_{i=Y+1}^{Z} iN(i) = (Z - Y)(Y + 1)N(Y + 1).$$

Together, we get that

$$(Y + 1)N(Y + 1) = \frac{T/Z}{\ln(Z/Y)} = \frac{U}{Z - Y},$$

or

$$\alpha = \frac{U}{T} = \frac{Z - Y}{Z \ln(Z/Y)} = \frac{Y/Z - 1}{\ln(Y/Z)} = \frac{\alpha' - 1}{\ln(\alpha')}.$$

where $\alpha' = Y/Z$, and is given by $\alpha' = -\alpha \cdot W(-\frac{1}{\alpha}e^{-1/\alpha}).$

The definition of the Lambert $W$ function states that if $z = xe^x$ then $x = W(z)$. If $\alpha = \frac{\alpha' - 1}{\ln(\alpha')}$, then

$\alpha \ln(\alpha') = \alpha' - 1$ and $\alpha' = e^{\frac{\alpha'}{\alpha} - \frac{1}{\alpha}}$, or $(\frac{\alpha'}{\alpha})e^{(-\frac{\alpha'}{\alpha})} = -\frac{1}{\alpha}e^{-\frac{1}{\alpha}}$. Thus, we get that $-\frac{\alpha'}{\alpha} = W(-\frac{1}{\alpha}e^{-\frac{1}{\alpha}})$ and $\alpha' = -\alpha \cdot W(-\frac{1}{\alpha}e^{-\frac{1}{\alpha}})$.

Now, we deduce that for every $Z - Y$ logical page writes, $Z$ physical pages are written. Hence, $P = L \cdot \frac{Z}{Z - Y} = \frac{L}{1 - \alpha'},$ and

$$E = \frac{P}{Z} = \frac{L}{Z(1 - \alpha')}, \quad EF_G(\alpha) = \frac{E}{L/Z} = \frac{1}{1 - \alpha'} = \frac{1}{1 + \alpha \cdot W(-\frac{1}{\alpha}e^{-\frac{1}{\alpha}})}.$$

Assume pages are written uniformly at random and now a least recently used (LRU) garbage collection policy is implemented (instead of the greedy garbage collection). Under this policy the blocks are written and erased sequentially. That is, first all blocks are written sequentially. Then the first block is erased while its valid pages are copied and written again into this block, then the second block and so on until the last block, and this process continues this way.

**Problem 1.** Find the connection between the erasure factor (write amplification) and overprovisioning (storage rate) under the LRU policy. That is, find the value of $EF_{LRU}(\alpha)$.

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3While there are better approximations to the differences between two Harmonic series, we choose this one since it provides better expressions which can be analyzed without dependency on the number of pages in a block.
Solution: The main problem is to find the expected number of valid pages when a block is erased. For a given block, after it is written, the number of valid pages is $Z$. When it is erased for garbage collection, the number of physical pages that were written is $T$. If the erasure factor $EF$ is some value $A$, then writing $T$ physical pages corresponds to writing $T/A$ logical pages. Thus, the probability that a logical page in this block will be still valid after writing these $T$ physical pages, or $T/A$ logical pages, is equal to (when $U$ is large enough)

$$
\left(1 - \frac{1}{U}\right)^\frac{T}{A} = \left(1 - \frac{1}{U}\right)^{\frac{U}{A}} = e^{-\frac{1}{\alpha A}}.
$$

Thus, the expected number of valid pages when erasing a block is $Y = e^{-\frac{1}{\alpha A}} Z$. The connection between the erasure factor $EF = A$ and $Y$ is given by $A = \frac{Z}{Z-Y}$, and therefore we get that

$$
A = \frac{1}{1 - e^{-\frac{1}{\alpha A}}}.
$$

Let us denote $t = \frac{1}{\alpha A}$, so we get

$$
\frac{1}{t \alpha} = \frac{1}{1 - e^{-t}}.
$$

Thus, $(1 - t \alpha) \cdot e^t = 1$ and

$$
\ln(1 - t \alpha) + t = 0.
$$

Now, denote $t' = 1 - t \alpha$ (that is, $t = \frac{1-t'}{\alpha}$), so we get

$$
\ln(t') + \frac{1-t'}{\alpha} = 0
$$

and

$$
\ln(t') - \frac{t'}{\alpha} = -\frac{1}{\alpha},
$$

so

$$
t' e^{-\frac{t'}{\alpha}} = e^{-\frac{1}{\alpha}}.
$$

Therefore, $-\frac{t'}{\alpha} e^{-\frac{t'}{\alpha}} = -\frac{1}{\alpha} e^{-\frac{1}{\alpha}}$ and $-\frac{t'}{\alpha} = W\left(-\frac{1}{\alpha} e^{-\frac{1}{\alpha}}\right)$, or $t' = -\alpha \cdot W\left(-\frac{1}{\alpha} e^{-\frac{1}{\alpha}}\right)$. Finally, we conclude that

$$
A = EF_{LRU}(\alpha) = \frac{1}{\alpha t} = \frac{1}{1 - t'} = \frac{1}{1 + \alpha \cdot W\left(-\frac{1}{\alpha} e^{-\frac{1}{\alpha}}\right)}.
$$