7. H. Berens and G. G. Lorentz, Theorems of Korovkin type for positive linear operators on Banach lattices, Approximation Theory (Ed. G. G. Lorentz), Academic Press, New York, 1973.

8. S. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul des probabilites. Comm. Soc. Math. Kharkow, (2) 13 (1912) 1–2.

9. G. Choquet, Lectures on Analysis, Vol. II, Representation Theory, Benjamin, New York, 1969.

10. K. Donner, Korovkin theorems for positive linear operators, J. Approximation Theory, 13 (1975) 443–450.

11. ———, Korovkin theorems and *P*-essential sets, to appear in J. Approximation Theory.

12. M. W. Grossman, Korovkin theorems for adapted spaces with respect to a positive operator, Math. Ann., 220 (1976) 253–262.

13. E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer-Verlag, New York, 1965.

14. K. Jacobs, Extremalpunkte konvexer Mengen, Selecta Mathematica III, Springer-Verlag, New York, 1971.

15. P. P. Korovkin, On convergence of linear positive operators in the space of continuous functions, Dokl. Akad. Nauk SSSR (N.S.), 90 (1953) 961–964 (Russian).

16. ———, Linear Operators and Approximation Theory, Hindustan Publ. Corp., Delhi, India, 1960.

17. S. S. Kutateladze, Choquet boundaries in *K*-spaces, Uspehi Mat. Nauk., 30 (1975) 107–146 (Russian). Translated in: Russian Math. Surveys, 30 (1975) 115–155.

18. G. Leha, Relative Korovkin-Sätze und Ränder, Math. Ann., 229 (1977) 87–95 and 233 (1978) 273–274.

19. R. R. Phelps, Lectures on Choquet's Theorem, Van Nostrand Math. Studies, 7, 1966.

20. Yu. A. Šaškin, Korovkin systems in spaces of continuous functions, Izv. Akad. Nauk SSSR, Ser. Mat., 26 (1962) 495–512 (Russian). Translated in: Amer. Math. Soc. Transl., Ser. 2, 54 (1966) 125–144.

21. ———, On the convergence of contractive operators, Mathematica (Cluj), 11 (34) (1969) 355–360 (Russian).

22. ———, On the convergence of linear operators, Konf. Konstruktiv. Teor. Funkciĭ, Varna 1970 (1972) 119–125 (Russian).

23. ———, Abstract harmonic functions and the problem of operator convergence. Mathematica (Cluj), 15 (38) (1973) 143–148 (Russian).

24. F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

25. M. Wolff, Uber die Korovkinhülle von Teilmengen in lokalkonvexen Vektorverbänden, Math. Ann., 213 (1975) 97–108.

26. ———, On Korovkin-type theorems in special function lattices, J. Approximation Theory (to appear).

27. D. E. Wulbert, Convergence of operators and Korovkin's theorem, J. Approximation Theory, 1 (1968) 381–390.

Mathematisches Institut der Universität Erlangen-Nürnberg, D-8520 Erlangen, West Germany.

ON THE DIVISION OF THE PLANE BY LINES

JOHN E. WETZEL

1. It occasionally happens in mathematics that the "real" reasons for the correctness of a result are quite unrelated to the arguments that comprise its proof. The reasoning used in proofs is severely restricted: the arguments must be carefully organized, logically correct, and complete. Every possibility has to be examined. The intuition is freer: intuitive insights may be fragmented, incomplete; rigor and logic are irrelevant; and even incorrect insights may sometimes be useful. While a good understanding of the proof of a result usually contributes to the full comprehension of the result, good intuitive insights may contribute even more. Indeed, appropriate intuitions can make an otherwise mysterious result seem virtually obvious.

One proof of a result may be preferred to another for many different reasons, some of them quite

---

John E. Wetzel received his Stanford Ph.D. in 1964 under H. L. Royden. He has been at the University of Illinois at Urbana-Champaign since 1961. By training an analyst, he is currently interested in partition problems in Euclidean and projective spaces.—*Editors* subjective. The preferred proof may be shorter, more surprising, more elegant; more (or less) elementary, more (or less) computational, more (or less) abstract. It may require less (or more) prior knowledge. It may be special, in the sense that it works only for the problem at hand, or it might be an instance of a more general method. Another, less obvious reason for preferring one proof over another lies in its accessibility to the intuition. The nearer the proof is, in some sense, to the intuition, the easier it may be to grasp.

In this note we illustrate this situation by giving three quite different elementary proofs of a little-known formula for the number of regions formed by an arbitrary arrangement of lines in the plane, a formula that was given in 1889 by Samuel Roberts **[15]** and rediscovered in 1963 by Albert Blank **[5]**. In section 2 we use some pretty heuristic arguments originally advanced by Roberts to find the formula. In the sections that follow we give first an unenlightening proof by mathematical induction, then an easy but *ad hoc* argument for counting the regions directly, and finally an intuitively appealing, elegant argument that is based on the notion of a sweep-line.

2. Suppose first of all that we have n lines in "general position," that is to say, n lines so arranged that each two meet in a point and no three pass through the same point. J. Steiner **[16]** proved in 1826 that n lines in general position divide the plane into

$$R=1+n+\binom{n}{2} \tag{1}$$

regions, of which

$$R'=1-n+\binom{n}{2}=\binom{n-1}{2} \tag{2}$$

are bounded. These formulas are easily proved by recursion, and we take them as known. They are discussed from the heuristics point of view by G. Pólya in [**14**, pp. 43–52 and problems 11, 15, 16, p. 54 (solutions, pp. 223, 224)]. A proof without frills appears in Golovina and Yaglom [**8**, p. 83].

Lines in the plane can fail to be in general position in two different ways: there may be more than two lines through a point, and there may be parallels. Both kinds of degeneracies reduce the number of points of intersection, because the intersection points coincide at multiple points, and parallel lines do not intersect at all. It is plain that regions and points of intersection are somehow closely related to each other—a point is produced when lines come together to pinch off a region—but the precise relationship is elusive.

Let us look more closely at each of these two kinds of degeneracies. First consider a multiple point M of multiplicity λ. We imagine the λ lines through M displaced a little (Fig. 1) to make an arrangement of λ lines in general position. According to (2), these λ lines in general position form

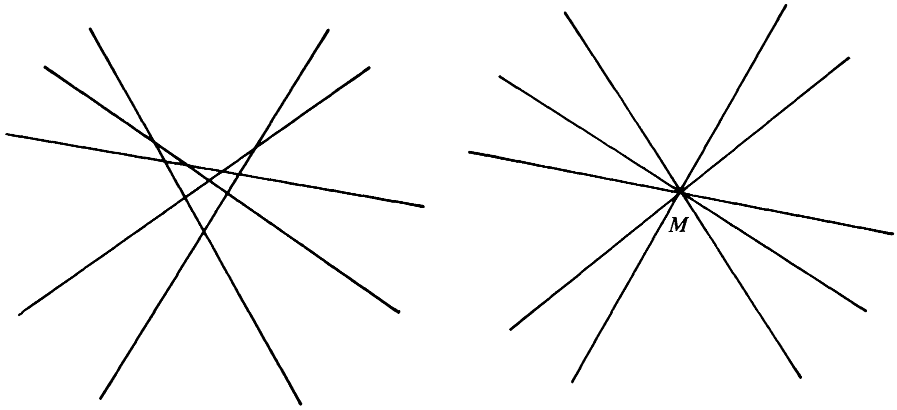

FIG. 1. The generation of a multiple point

$\binom{\lambda-1}{2}$ bounded regions, and all are lost when the lines are brought again to concurrency. So the concurrency of λ lines at M clearly causes the loss of $\binom{\lambda-1}{2}$ regions.

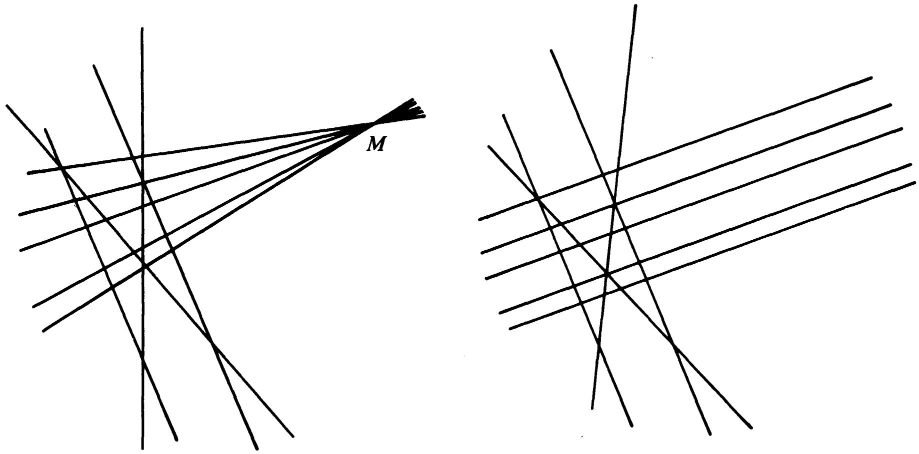

FIG. 2. The generation of a family of parallels

Now consider a family of μ parallel lines in a direction d. We imagine the lines displaced a little to pass through a common point M far away (Fig. 2), and we rebuild the parallels by letting M tend to infinity in the direction d. Then $\binom{\mu-1}{2}$ regions are lost at the point M, and $\mu-1$ further regions are lost beyond M. So the total loss due to the μ parallels is $\binom{\mu-1}{2}+(\mu-1)=\binom{\mu}{2}$.

If there are m multiple points $M_1, M_2, \ldots, M_m$ in the given arrangement of n lines with, say, $\lambda_i \geqslant 3$ lines passing through M_i, then the total loss of regions ought to be

$$\binom{\lambda_1-1}{2}+\binom{\lambda_2-1}{2}+\cdots+\binom{\lambda_m-1}{2}$$

regions; and if there are p parallel families with, say, $\mu_j \geqslant 2$ lines in the jth family, the total loss of regions due to the parallels ought to be

$$\binom{\mu_1}{2}+\binom{\mu_2}{2}+\cdots+\binom{\mu_p}{2}$$

regions. Hence one must have

$$R=1+n+\binom{n}{2}-\sum_{i=1}^{m}\binom{\lambda_i-1}{2}-\sum_{j=1}^{p}\binom{\mu_j}{2}. \tag{3}$$

This is Roberts' formula. It gives R as "the number of regions formed by n lines in general position" minus "the number of regions lost because of the multiple points" minus "the number of regions lost because of the parallels."

Roberts' heuristic arguments for (3) are convincing, not to say compelling. They make the formula seem almost obvious. Figure 3 pictures an arrangement of $n=12$ lines that meet to form four multiple points of multiplicities 3, 3, 3, and 4 and lie in parallel families containing 2, 3, and 5 lines. Both the formula and a visual count give $R=59$, so the formula is correct for this fairly complicated arrangement. Surely it must be correct!

And yet, these arguments do not immediately yield a correct proof. The difficulty is that moving one line to destroy a multiple point or a parallel family may significantly alter the arrangement in

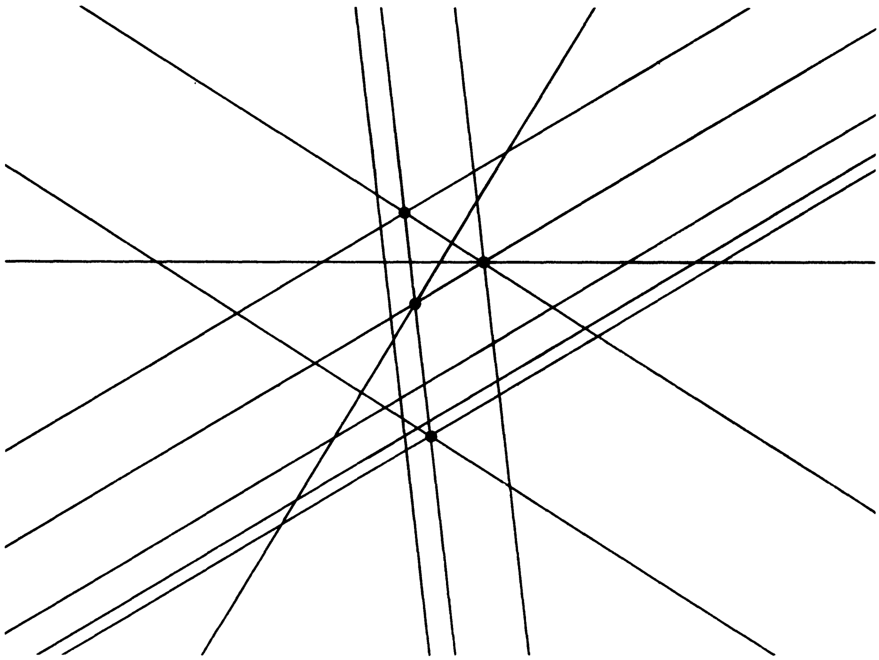

FIG. 3. An arrangement of 12 lines that divides the plane into 59 regions

some other way. Consider, for example, a line that passes through three collinear multiple points and belongs to a parallel family. Moving that line so as to destroy one of the multiple points necessarily changes at least one other multiple point, and new regions are produced if the line is turned. You cannot wiggle a part of a line without moving the whole line, and the prospect of keeping track of all the changes that might occur appears to be an overwhelming notational chore.

Nevertheless, the heuristic arguments are very persuasive. The formula surely cannot be false, and indeed it is not.

3. As our first proof of the formula we give a concise and correct argument by mathematical induction. To carry this proof out we need to introduce some notation.

Let α be an arrangement of n lines. For each point P of the plane, write $\lambda(P)$ for the number of lines of α that pass through P. We call P a *simple* point of α if $\lambda(P)=2$ and a *multiple* point of α if $\lambda(P)\geqslant 3$. For each direction d, write $\mu(d)$ for the number of lines of α that lie in the direction d. We call d a *multiple* direction if $\mu(d)\geqslant 2$.

If we adopt the usual combinatorial convention that $\binom{a}{b}=0$ when $b>a$, we can write

$$\sum_{i=1}^{m}\binom{\lambda_i-1}{2}=\sum_{P}\binom{\lambda(P)-1}{2}$$

and

$$\sum_{j=1}^{p}\binom{\mu_j}{2}=\sum_{d}\binom{\mu(d)}{2},$$

because $\binom{\lambda(P)-1}{2}=0$ unless P is a multiple point, and $\binom{\mu(d)}{2}=0$ unless d is a multiple direction. In this notation, Roberts' formula (3) becomes

$$R=1+n+\binom{n}{2}-\sum_{P}\binom{\lambda(P)-1}{2}-\sum_{d}\binom{\mu(d)}{2}. \tag{4}$$

Formula (4) can easily be verified for $n=0$ and 1 and also for the case in which all n lines are parallel. Suppose that (4) holds for every arrangement of n lines. Take any arrangement α' of $n+1$ lines not all of which are parallel, and denote its counters by λ' and μ'. Select a line L in α' and delete it, leaving an arrangement α of n lines for which, by the induction hypothesis, formula (4) holds. We restore L to rebuild α' from α and watch closely what happens.

If the chosen line L lies in the direction d_0, then the counters for α' are related to those of α by the formulas

$$\lambda'(P)=\begin{cases}\lambda(P) & \text{if } P\notin L\\ \lambda(P)+1 & \text{if } P\in L,\end{cases}$$

$$\mu'(d)=\begin{cases}\mu(d) & \text{if } d\neq d_0\\ \mu(d_0)+1 & \text{if } d=d_0.\end{cases}$$

A moment's thought shows that when L is restored, it meets the lines of α in exactly

$$N=n-\mu(d_0)-\sum_{P\in L}\binom{\lambda(P)-1}{1}\geqslant 1$$

points, for L does not cross $\mu(d_0)$ of the n lines of α, and a point of concurrency of $\lambda(P)$ lines adds only one point of intersection. These N points partition L into two rays and $N-1$ line segments, and each of these $N+1$ pieces divides a region of α into two parts, producing exactly $N+1$ new regions. Consequently there must be exactly

$$\Delta R=n+1-\sum_{P\in L}\binom{\lambda(P)-1}{1}-\mu(d_0) \tag{5}$$

new regions created when L is restored.

The two sums over points in (4) and (5) combine to give

$$\sum_{P\in L}\left[\binom{\lambda(P)-1}{2}+\binom{\lambda(P)-1}{1}\right]+\sum_{P\notin L}\binom{\lambda(P)-1}{2}$$

$$=\sum_{P\in L}\binom{\lambda(P)}{2}+\sum_{P\notin L}\binom{\lambda(P)-1}{2}$$

$$=\sum_{P}\binom{\lambda'(P)-1}{2}.$$

Similarly, the two sums over directions give

$$\left[\binom{\mu(d_0)}{2}+\binom{\mu(d_0)}{1}\right]+\sum_{d\neq d_0}\binom{\mu(d)}{2}$$

$$=\binom{\mu(d_0)+1}{2}+\sum_{d\neq d_0}\binom{\mu(d)}{2}$$

$$=\sum_{d}\binom{\mu'(d)}{2}.$$

It follows that α' defines exactly

$$\begin{aligned}R'&=R+\Delta R\\ &=1+(n+1)+\binom{n+1}{2}-\sum_{P}\binom{\lambda'(P)-1}{2}-\sum_{d}\binom{\mu'(d)}{2},\end{aligned}$$

which is Roberts' formula for α'. This completes the proof by induction.

This argument is correct, complete, logical, and not very enlightening. The key observation, that the *new* regions are in one-one correspondence with the segments and rays formed on the new line by the points in which it meets the lines already in place, is buried in the middle. The mechanics of the transition are difficult to see through; the formula appears to come from a fortuitous combination of binomial coefficients that could hardly have been foreseen. Indeed, this proof contributes little more to our understanding of the formula than the fact, important though it is, that it is correct.

4. The intuitive content of Roberts' formula is rich and varied, and different insights can be found that lead to correct and elegant proofs. We develop one such argument next.

Our first observation is that translating a line of the arrangement does not alter the structure of the parallel families. What does it do to the region count?

If the line L to be translated initially does not pass through any multiple points, and if in its final resting position it does not pass through any multiple points, we claim that the region count is unaltered by the translation. Indeed, it is clear that the region count could change during the transition only when L slides through a point of intersection; and at each such point P precisely as many new regions are created when L leaves P as were lost when L came to P, viz., $\lambda(P)-1$.

Now we ask what happens when we tear down a multiple point by translating its lines away one by one. If λ lines pass through the multiple point M, the first line to go creates $\lambda-2$ new regions having M as a boundary point; then the next line to leave produces $\lambda-3$ more new regions; and so forth, until the point is reduced to a simple point. In the process, exactly

$$(\lambda-2)+(\lambda-3)+\cdots+2+1=\binom{\lambda-1}{2}$$

new regions are produced from the multiple point M. Note that if all of the multiple points of the arrangement are destroyed in this way, the fact that some of them may be collinear is immaterial.

Given an arbitrary arrangement α of n lines having m multiple points $M_1,M_2,\ldots,M_m$ of multiplicity $\lambda_1,\lambda_2,\ldots,\lambda_m$ and forming R regions, tear down all of its multiple points by making small translations of its lines. This procedure evidently produces a new arrangement α' having no multiple points and having

$$R'=R+\sum_{i=1}^{m}\binom{\lambda_i-1}{2} \tag{6}$$

regions.

Suppose the lines of the original arrangement α (and therefore those of the new arrangement α') fall into s parallel families of $n_1,n_2,\ldots,n_s$ lines, respectively, where to accommodate lines having no parallel partners we allow n_i to be 1. If there are p multiple directions (as in the notation of section 2), then the n_k sum to n, each μ_j is an n_k, and there are $s-p\geqslant 0$ lines that have no parallel partners. Unless the lines are all parallel, in which case Roberts' formula is trivially true, we select one line from each parallel family to obtain an arrangement of s lines in general position. By translating the other lines to new positions near their chosen partners (being careful not to introduce multiple points), we can arrange the lines in s "narrow" parallel families without changing the region count; and indeed we can make the families so narrow (Fig. 4) that no points of intersection come between any two parallels. For lines so arranged, we can count the regions by inspection.

First of all, there are clearly $(n_i-1)(n_j-1)$ little parallelograms formed where the n_i parallel lines in the ith family cross the n_j parallel lines in the jth family. (These regions are heavily shaded in Figure 4.) Summing, we find there are

$$\sum_{i<j}(n_i-1)(n_j-1)=\sigma_2-(s-1)n+\binom{s}{2}$$

regions of this kind, where σ_2 is the sum of the $\binom{s}{2}$ products n_in_j with $1\leqslant i<j\leqslant s$.

There are evidently $s(n_i-1)$ further regions formed between the n_i lines in the ith family, because there are $s-1$ other intersecting families. So in all there are

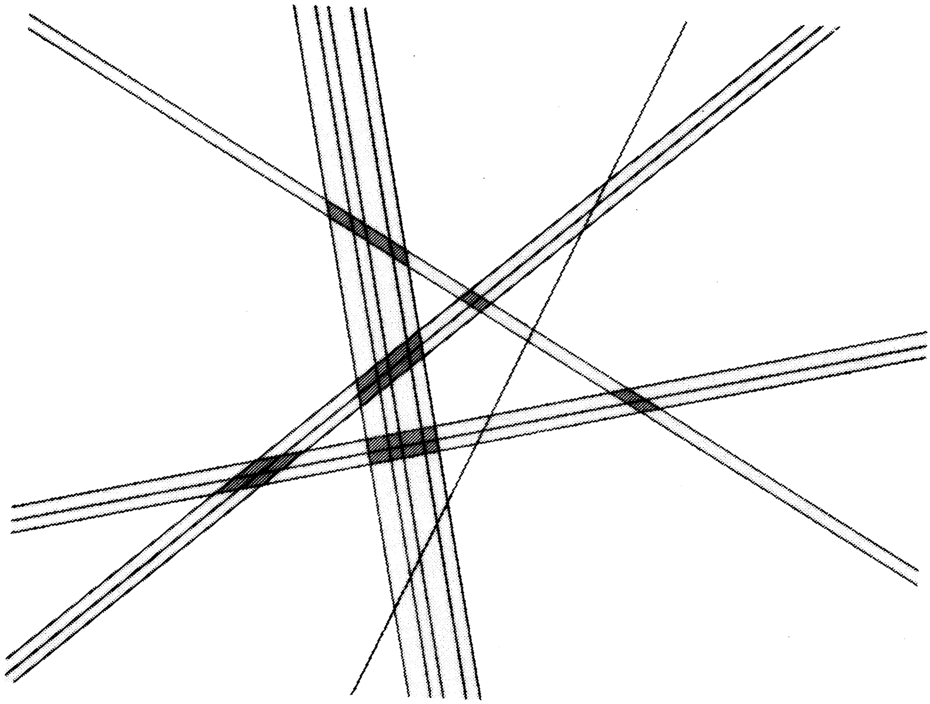

FIG. 4. An arrangement of 14 lines in 5 narrow parallel families

$$\sum_{i=1}^{s} s(n_i-1)=sn-s^2$$

regions of this sort (lightly shaded in Figure 4).

The same number of regions remain as are formed by s lines in general position, viz., $1+s+\binom{s}{2}$. So in all there are

$$\begin{aligned} R' &= 1+s+\binom{s}{2}+sn-s^2+\sigma_2-(s-1)n+\binom{s}{2} \\ &= 1+n+\sigma_2 \end{aligned} \tag{7}$$

regions in the arrangement α'. This pretty formula was proved (by recursion) by J. Steiner in 1826.

Using the easy identity

$$\sigma_2=\binom{n_1+n_2+\cdots+n_s}{2}-\sum_{t=1}^{s}\binom{n_t}{2},$$

which can be verified by simply expanding out, we can rewrite (7) in Roberts' form:

$$R'=1+n+\binom{n}{2}-\sum_{t=1}^{s}\binom{n_t}{2}=1+n+\binom{n}{2}-\sum_{j=1}^{p}\binom{\mu_j}{2}.$$

Roberts' formula for the original arrangement α now follows immediately from (6):

$$\begin{aligned} R &= R'-\sum_{i=1}^{m}\binom{\lambda_i-1}{2} \\ &= 1+n+\binom{n}{2}-\sum_{i=1}^{m}\binom{\lambda_i-1}{2}-\sum_{j=1}^{p}\binom{\mu_j}{2}. \end{aligned}$$

Although this argument is fairly natural, direct, and easy to follow, it seems somehow quite special.

There is little hint of a general method. If Steiner's formula (7) is a known prior result, then the deduction of Roberts' formula from Steiner's is short and elegant. Unfortunately, Steiner's formula is itself not very well known.

5. Finally we present a proof that is based on the ingenious notion of a sweep-line introduced in 1955 by H. Hadwiger **[10]** and developed by A. Brousseau **[6]** in a pretty paper concerned with the heuristics of partition problems. This argument is elegant, transparent, and intuitive. It throws considerable light on Roberts' formula and provides intuitive insights quite different from those given by Roberts' heuristic arguments and from those of the previous section. It has the further advantage that no prior knowledge of special cases need be assumed.

Take a line b, which with Brousseau we call a sweep-line, not parallel to any of the lines of the given arrangement and initially located so far away that all of the points of intersection of the arrangement are on the same side. In this initial position, b is divided into two rays and $n-1$ segments by the n points of intersection with the n given lines, and each of these segments and rays lies in (and so counts) a well-defined region. So the sweep-line initially identifies $1+n$ regions.

Now sweep the line b across the arrangement, moving it always parallel to its initial position. New regions are encountered precisely at the points in which the given lines intersect—one new region at each simple point, two at each point of multiplicity three, and in general, $\lambda-1$ new regions at each point of multiplicity λ. Suppose there are S simple points. Then the sweep-line encounters exactly

$$S+\sum_{i=1}^{m}(\lambda_i-1)$$

new regions during its transit across the plane, and since every region is eventually counted, exactly

$$R=1+n+S+\sum_{i=1}^{m}(\lambda_i-1) \tag{8}$$

regions are formed by the given lines.

A formula for the number S of simple points may be deduced from the obvious fact that each two lines either intersect or are parallel. If $\lambda_i \geqslant 3$ lines pass through M_i, there evidently are $\binom{\lambda_i}{2}$ pairs of lines that intersect at M_i. Consequently there are, in all,

$$k_1=S+\sum_{i=1}^{m}\binom{\lambda_i}{2}$$

intersecting pairs of lines in the arrangement.

If there are $\mu_j \geqslant 2$ parallel lines in the jth parallel family, then there plainly are $\binom{\mu_j}{2}$ pairs of lines in this family that do not meet. Consequently there are, in all,

$$k_2=\sum_{j=1}^{p}\binom{\mu_j}{2}$$

pairs of non-intersecting lines in the arrangement.

Since each of the $\binom{n}{2}$ pairs of lines in the arrangement is either intersecting or parallel, we must have $\binom{n}{2}=k_1+k_2$. The desired formula for S follows:

$$S=\binom{n}{2}-\sum_{i=1}^{m}\binom{\lambda_i}{2}-\sum_{j=1}^{p}\binom{\mu_j}{2}.$$

Substituting this formula for S into (8), we find that

$$R=1+n+\binom{n}{2}-\sum_{i=1}^{m}\left[\binom{\lambda_i}{2}-\lambda_i+1\right]-\sum_{j=1}^{p}\binom{\mu_j}{2},$$

which, in view of the identity

$$\binom{\lambda_i}{2}-\lambda_i+1=\binom{\lambda_i-1}{2},$$

is Roberts' formula.

The sweep-line plays a mathematical role in the argument, to be sure, but it plays an interesting psychological role as well. With its aid we focus our attention on the plane one cross-section strip at a time and obtain the number R of regions as the sum of two parts: the first an initial contribution that is entirely independent of the details of the arrangement, the second the sum of "local" changes that occur precisely at the points of intersection. Watching the sweep-line as it passes through a point of intersection, we see exactly what the change in the region count really is at that point, and why. This makes some precise sense of the observation that regions are somehow closely related to points of intersection.

An important aspect of the sweep-line method is that one need not know the result in advance, because the argument allows the result to be discovered. This makes the sweep method an important heuristic technique for plane partition problems. It generalizes in a natural way to ovals, to higher dimensional Euclidean spaces, and to projective spaces. The sweep idea is surely one of the most ingenious recent ideas of elementary mathematics.

6. Brousseau's pretty formula (8) can also be deduced from Euler's formula $f=2-v+e$ for connected plane graphs together with some simple incidence relations. Similar considerations permit yet another proof of Roberts' formula (see Alexanderson and Wetzel [**2**]). A variant of this argument for arrangements of pseudolines (where one can, in fact, wiggle just a part of a line) appears in Alexanderson and Wetzel [**4**].

There is an extensive literature on partition problems in Euclidean and projective spaces. Grünbaum [**9**] summarizes much of what is known and includes a large bibliography. Other partition problems are discussed in Alexanderson and Wetzel [**2**] and [**3**], Freeman [**7**], and Kerr and Wetzel [**11**], [**12**], and [**13**]. A survey of partition formulas for three- and four-dimensional Euclidean and projective spaces is presented in Alexanderson and Wetzel [**1**].

An entirely different approach to problems of this kind is developed with great success by Zaslavsky in [**17**] and [**18**].

It is a pleasure to thank Marianne Jankowski for preparing the figures and to acknowledge many fruitful conversations on the ideas of this note with G. L. Alexanderson.

References

1. G. L. Alexanderson and John E. Wetzel, Arrangements of planes in space (in preparation).

2. ———, Dissections of a plane oval, this MONTHLY, 84 (1977) 442–449.

3. ———, Simple partitions of space, to appear in Math. Mag.

4. ———, Three m-point plane partition problems (in preparation).

5. Albert A. Blank, Mathematical induction, in Enrichment Mathematics for High School, 28th Yearbook, National Council of Teachers of Mathematics, Washington, D.C., 1963, 118–140.

6. Bro. U. Alfred (Brousseau), A mathematician's progress, Math. Teacher, 59 (1966) 722–727.

7. J. W. Freeman, The number of regions determined by a convex polygon, Math. Mag., 49 (1976) 23–25.

8. L. I. Golovina and I. M. Yaglom, Induction in Geometry (trans. by A. W. Goodman and Olga A. Titelbaum), Heath, Boston, 1963.

9. Branko Grünbaum, Arrangements and Spreads, Conference Board of the Mathematical Sciences, Regional Conference Series in Math. No. 10, Amer. Math. Soc., Providence, R.I., 1972.

10. H. Hadwiger, Eulers Charakteristik und kombinatorische Geometrie, J. Reine Angew. Math., 194 (1955) 101–110.

11. Jeanne W. Kerr and John E. Wetzel, Dissections of a polygon, Elem. Math., 33 (1978) 1–7.

12. ———, Dissections of a triangular prism, Geometriae Dedicata, 4 (1975) 279–289.

13. ———, Platonic divisions of space, to appear in Math. Mag.

14. G. Pólya, Induction and Analogy in Mathematics, Vol. 1 of Mathematics and Plausible Reasoning, Princeton Univ. Press, 1954.

15. S. Roberts, On the figures formed by the intercepts of a system of straight lines in a plane, and on analogous relations in space of three dimensions, Proc. London Math. Soc., 19 (1889) 405–422.

16. J. Steiner, Einige Gesetze über die Theilung der Ebene und des Raumes, J. Reine Angew. Math., 1 (1826) 349–364.

17. Thomas Zaslavsky, Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes, Mem. Amer. Math. Soc. No. 154, Amer. Math. Soc., Providence, R.I., 1975.

18. ———, A combinatorial analysis of topological dissections, Advances in Math., 25 (1978) 267–285.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801.

CORRECTIONS TO
"Extended Mean Values"

(THIS MONTHLY, 85 (1978) 84–90)

E. B. LEACH AND M. C. SHOLANDER

Due to mail malhandling on the southern shore of the Mediterranean, final corrections on galley sheets were not made in time for paper publication.

1. Table (3.1). Row 3, column 2, entry should be "$\sin w$". Row 4, column 3, entry should be "$(\tanh w)/w$".

2. To each of (3.8), (3.9), and (3.11) should be added the restriction $x \neq y$. In the proof of (3.11), parentheses around a fraction numerator are missing.

3. In the line sixth from the last, the sector should extend from $-\pi/4$ to $\pi/2$ radians.

4. In paragraph 1 of Section 1, it was intended that N be described as the "root-mean square." The authors are aware that statisticians use "root mean square" when referring to the root mean-square M_2.

5. In the introduction, it is mentioned that Tobey (in 1967) described means which had extended means as special cases. We are informed that similar credit should be given to H. Brøns' 1967 lecture notes at the Institute for Mathematical Statistics in Copenhagen. Following these notes, we have in Ann. Math. Stat. 40 (1969) 339–355 a paper "Generalized Means and Associated Families of Distributions" by H. Brøns, H. D. Brunk, W. Franck, and D. L. Hanson where, after such means are defined, various statistical applications are made.

DEPARTMENT OF MATHEMATICS AND STATISTICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OH 44106.

MISCELLANEA

13. Analysis takes back with one hand what it gives us with the other. I recoil with fear and loathing from that deplorable evil, continuous functions with no derivatives.

Hermite to Stieltjes, 20 May 1893.

(Note: Weierstrass's example of a continuous nowhere differentiable function had been published (by du Bois-Reymond) 18 years earlier.)