Introduction to Network Coding, Bounds and Constructions

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Lecture 4

Algebraic Approach for Network Coding
Algebraic Approach

Outline

- Algebraic representation for network coding
- Multicasting is solvable with network coding
- Linear information flow algorithm
- Random network coding
Multicasting in Network Coding

Find two edge disjoint paths from the source $S$ to each receiver.

Fragouli, Soljanin 2006
Multicasting in Network Coding

Two edge disjoint paths from the source $S$ to each receiver.
Multicasting in Network Coding
Multicasting in Network Coding

\[ x, y \]

\[ S \]

\[ A \]

\[ B \]

\[ C \]

\[ D \]

\[ E \]

\[ F \]

\[ R_1 \]

\[ R_2 \]

\[ R_3 \]
Multicasting in Network Coding

\[
\begin{align*}
A & \xrightarrow{x} B \\
B & \xrightarrow{x} C \\
A & \xrightarrow{x} D \\
B & \xrightarrow{x} E \\
E & \xrightarrow{\alpha_1 x + \alpha_2 y} F \\
C & \xrightarrow{y} D \\
B & \xrightarrow{y} E \\
E & \xrightarrow{\alpha_1 x + \alpha_2 y} F \\
F & \xrightarrow{\alpha_3 x + \alpha_4 \alpha_1 x + \alpha_2 y} R_3 \\
S & \xrightarrow{x, y} A \\
S & \xrightarrow{x} R_1 \\
R_1 & \xrightarrow{\alpha_1, \alpha_2} [0, 1] \\
R_2 & \xrightarrow{1, 0} [\alpha_1 + \alpha_2 \alpha_4, \alpha_2 \alpha_4] \\
R_3 & \xrightarrow{\alpha_1, \alpha_2} [\alpha_1, \alpha_2] \\
\end{align*}
\]
The source $S$ has $h$ messages, $X^{tr} = (x_1, x_2, \ldots, x_h)$. There are $N$ receivers $R_1, R_2, \ldots, R_N$ each one demands all the $h$ messages.

Can all the messages be received simultaneously by receiver $R_j$?

Yes, if the min-cut between $S$ and $R_j$ has size at least $h$.

OR

There are $h$ edge-disjoint paths between $S$ and $R_j$. 
Can all the messages be received simultaneously by all receivers?

Yes, if each node can re-encode the information.

Edges carry linear combinations of their parent node inputs, where \( \{\alpha_i\} \) are the coefficients used in these linear combinations.

Edges carry linear combinations of the \( h \) messages.
Edges carry linear combinations of their parent node inputs, where $\{\alpha_i\}$ are the coefficients used in these linear combinations.

The coefficients of the given edge form the local coding vector.

Edges carry linear combinations of the $h$ messages.

The coefficients of these linear combinations form the global coding vector.
Let $y^j_i$ be the symbol on the last edge in the $i$-th path from $S$ to the receiver $R_j$.

Let $c_1(e), c_2(e), \ldots, c_h(e)$ be the coefficients of $x_1, x_2, \ldots, x_h$ in the linear combination on the edge $e$, i.e., if $y$ is the symbol computed on the edge $e$ then

$$y = c_1(e) \cdot x_1 + c_2(e) \cdot x_2 + \cdots + c_h(e) \cdot x_h$$

or

$$y = (c_1(e), c_2(e), \ldots, c_h(e)) \cdot X.$$
Let $c^j_i$ be the global coding vector on the last edge in the $i$-th path from $S$ to the receiver $R_j$.

Let $C_j$ be the $h \times h$ matrix whose $i$th row is $c^j_i$.

$C_j$ is the transfer matrix of receiver $R_j$.

Let $y^j_i$ be the symbol computed on the last edge in the path from node $S_i$ to the receiver $R_j$.

Receiver $R_j$ has to solve the system of equations

$$Y_j = C_j \cdot X,$$

where $Y_j = (y^j_1, y^j_2, \ldots, y^j_h)^{tr}$. 
Receiver $R_j$ has to solve the system of equations

$$Y_j = C_j \cdot X,$$

where $Y_j = (y^j_1, y^j_2, ..., y^j_h)^{tr}$.

How can we make sure that each receiver will compute the right messages?

The matrices $C_1, C_2, ..., C_N$ must be of full rank.

Select coefficients $\{\alpha_i\}$ such that

$$f(\{\alpha_i\}) \triangleq \det(C_1) \cdot \det(C_2) \cdots \det(C_N) \neq 0.$$
Let \( f(z_1, z_2, \ldots, z_\eta) \) be a polynomial over \( \mathbb{F}_q \) such that the maximum degree of each variable in a term of \( f(z_1, z_2, \ldots, z_\eta) \) is at most \( d \). Let \( A \) be a set of \( d + 1 \) distinct elements of \( \mathbb{F}_q \).

If \( f(a_1, a_2, \ldots, a_\eta) = 0 \) for all \( \eta \)-tuples in \( A^n \), then \( f \) is identically the zero polynomial.
The proof is by induction on $\eta$. For $\eta = 1$, $f$ is a polynomial in a single variable of degree at most $d$ and hence it can have at most $d$ zeros.

Suppose that for some $\eta \geq 1$ the claim is true for all such polynomials. It can be written as

$$f(z_1, \ldots, z_\eta, z_{\eta+1}) = \sum_{i=0}^{d} f_i(z_1, \ldots, z_\eta)z_{\eta+1}^i$$

where $f_i$ are polynomials with degrees of variables bounded by $d$.

Let $f(z_1, z_2, \ldots, z_\eta, z_{\eta+1})$ be such a polynomial.
**Code Design**

**Proof**

\[ f(z_1, \ldots, z_\eta, z_{\eta+1}) = \sum_{i=0}^{d} f_i(z_1, \ldots, z_\eta)z_{\eta+1}^i \]

where \( f_i \) are polynomials with degrees bounded by \( d \)

Suppose \( f(z_1, \ldots, z_\eta, z_{\eta+1}) = 0 \) for all \((\eta+1)\)-tuples in \( A^{\eta+1} \)

By the assumption each such \( f_i \) must be identically zero which implies that \( f \) is zero.

Then each polynomial \( f(a_1, \ldots, a_\eta, z_{\eta+1}) \) in \( z_{\eta+1} \) (fixing \((a_1, \ldots, a_\eta)\)) has at least \( d+1 \) zeros, and thus must be identically zero.

Therefore, \( f_i(a_1, \ldots, a_\eta) = 0 \) for all \((a_1, \ldots, a_\eta) \in A^{\eta}\)
Let $f(z_1, z_2, \ldots, z_\eta)$ be a polynomial over $\mathbb{F}_q$ such that

- $f(z_1, z_2, \ldots, z_\eta)$ is not identically zero;
- The maximum degree of each variable in a term of $f(z_1, z_2, \ldots, z_\eta)$ is at most $d$;
- $q > d$.

Then, there exist values $b_1, b_2, \ldots, b_\eta \in \mathbb{F}_q$ such that

$$f(b_1, b_2, \ldots, b_\eta) \neq 0.$$
$f(z_1, z_2, \ldots, z_\eta)$ is a polynomial over $F_q$, where

- $f(z_1, z_2, \ldots, z_\eta)$ is not identically zero;
- The sum of degrees of all the variables in a term of $f(z_1, z_2, \ldots, z_\eta)$ is at most $d$;
- $q > d$.

There exist values $b_1, b_2, \ldots, b_\eta \in F_q$ such that

$$f(b_1, b_2, \ldots, b_\eta) \neq 0.$$ 

**Why not any field?**

The polynomial

$$x(x - \alpha^0)(x - \alpha^1) \cdots (x - \alpha^{q-2}),$$

$\alpha$ primitive, evaluated to zero on all elements of $F_q$.

$x(x + 1) + x + x^2$ is identically zero over $F_2$. 
A multicast network is solvable if and only if the min-cut for each receiver is at least the number of messages.

A network with $h$ sources, where each one has exactly one message, is solvable if and only if the min-cut to each receiver is at least $h$. 

Theorem
Multicast in Undirected Network

The theorem does not hold for undirected graphs.
Introduction to Network Coding, Bounds and Constructions

A SHORT BREAK
A network \( G = (V, E) \) with \( h \) sources \( S_1, \ldots, S_h \in V \) and \( N \) receivers \( R_1, \ldots, R_N \in V \). Each source has one message and each receiver demands all the messages.

Remove each edge which will not disconnect a source-receiver pair. The new subgraph is a minimal multicast subgraph of \( G \); assume \( G \) is minimal.
Linear Information Flow Algorithm

Find $h$ edge-disjoint paths
\[ \{(S_i, R_j) : 1 \leq i \leq h\} \]
in the network $G$
to each receiver $R_j, 1 \leq j \leq N$.

A coding point is an edge where a path $(S_i, R_j)$
merges with $(S_\ell, R_m)$, where $i \neq \ell$ and $j \neq m$.

$i = \ell$ same information
on the paths

$j = m$ paths to receiver
are not edge-disjoint
Linear Information Flow Algorithm

Find $h$ edge-disjoint paths $\{ (S_i, R_j) : 1 \leq i \leq h \}$ in the network $G$ to each receiver.

Let $R(\delta)$ be the set of all receivers which have a path with the coding point $\delta$.

Each coding point $\delta$ appears in at most one path $(S_i, R_j)$ for $R_j$ ($h$ disjoint paths to $R_j$).

Let $f^j_\leftarrow(\delta)$ denote the predecessor coding point to $\delta$ along the path $(S_i, R_j)$. 
For \( R_j \) the algorithm maintains a set \( P_j \) of the last visited \( h \) coding points and a set \( B_j = \{ c^j_1, ..., c^j_h \} \) of the \( h \) global coding vectors.

Initially \( P_j \) contains the source nodes and \( B_j \) contains the unity vectors.

\( B_j \) must contain \( h \) linearly independent vectors.
The network is scanned in a way that an edge is scanned only after all the incoming edges of its parent node were scanned.

At step $k$, the algorithm assigns a coding vector $c(\delta_k)$ to the coding point $\delta_k$, and updates $P_j$ and $B_j$ for each $R_j \in R(\delta_k)$:

- the associated vector $c(f^j_\leftarrow(\delta_k))$ in $B_j$ with $c(\delta_k)$.
- the point $f^j_\leftarrow(\delta_k)$ in $P_j$ with the point $\delta_k$. 
The algorithm selects the vector \( c(\delta_k) \) in a way that for every receiver \( R_j \in R(\delta_k) \) the set \( (B_j \setminus \{c(f^j_\rightarrow(\delta_k))\}) \cup \{c(\delta_k)\} \) form a basis for the \( h \)-dimensional space.

Such a choice always exists provided that the field \( \mathbb{F}_q \) has size \( q \geq N \).

When the algorithm terminates \( B_j \) contains the set of linearly independent equations for \( R_j \).
Consider a coding point \( \delta \) with \( m \leq h \) parents and a receiver \( R_j \in R(\delta) \). Let \( V(\delta) \) be the \( m \)-dimensional subspace spanned by the coding vectors of the parents of \( \delta \), and \( V(R_j, \delta) \) be the \((h - 1)\)-dimensional subspace spanned by the elements of \( B_j \) after removing \( c(f_j^\perp(\delta)) \). Then
\[
\dim\{V(\delta) \cap V(R_j, \delta)\} = m - 1.
\]
**Lemma 1** Consider a coding point $\delta$ with $m \leq h$ parents and a receiver $R_j \in R(\delta)$. Let $V(\delta)$ be the $m$-dimensional subspace spanned by the coding vectors of the parents of $\delta$, and $V(R_j, \delta)$ be the $(h - 1)$-dimensional subspace spanned by the elements of $B_j$ after removing $c(f^I_j(\delta))$. Then

$$\dim (V(\delta) \cap V(R_j, \delta)) = m - 1.$$ 

**Proof**

$$\dim (V(\delta) \cap V(R_j, \delta)) = \dim V(\delta) + \dim V(R_j, \delta) - \dim (V(\delta) \cup V(R_j, \delta))$$

$$= m + h - 1 - \dim (V(\delta) \cup V(R_j, \delta))$$

$$\dim ((V(\delta) \cup V(R_j, \delta)) = h$$ since $V(\delta)$ contains $c(f^I_j(\delta))$ and $V(R_j, \delta)$ contains the rest of the basis for $B_j$. 
The algorithm successfully identifies a valid network code using any field $\mathbb{F}_q$ of size $q \geq N$.

**Lemma**

Consider a coding point $\delta$ with $m \leq h$ parents and a receiver $R_j \in R(\delta)$.

**Proof**

The coding vector $c(\delta)$ is a nonzero vector in the $m$-dimensional subspace $V(\delta)$ spanned by the coding vectors of the parents of $\delta$.

There are $q^m - 1$ vectors feasible for $c(\delta)$. 
The coding vector $c(\delta)$ is a nonzero vector in the $m$-dimensional subspace $V(\delta)$ spanned by the coding vectors of the parents of $\delta$.

There are $q^m - 1$ vectors feasible for $c(\delta)$.

If the network is solvable for receiver $R_j$, then $c(\delta)$ does not belong to the intersection of $V(\delta)$ and the $(h - 1)$-dimensional subspace $V(R_j, \delta)$ spanned by the elements of $B_j$ after $c \left( f^j_{\leftarrow}(\delta) \right)$ is removed.
The coding vector $c(\delta)$ has to be a nonzero vector in the $m$-dimensional subspace $V(\delta)$ spanned by the coding vectors of the parents of $\delta$. There are $q^m - 1$ such vectors, feasible for $c(\delta)$. If the network solvable for $R_j$, then $c(\delta)$ is not in the intersection of $V(\delta)$ and $V(R_j, \delta)$ spanned by the elements of $B_j$ after removing $c(f_j(\delta))$.

ByLemma 1, the dimension of this intersection is $m - 1$, and thus the number of vectors it excludes from $V(\delta)$ is $q^{m-1} - 1$.

The number of vectors excluded by the receivers in $R(\delta)$ is at most
\[ |R(\delta)| (q^{m-1} - 1) \leq N(q^{m-1} - 1) \]

Therefore, if $q^m - 1 > N(q^{m-1} - 1) \iff q \geq N$, then a valid value for $c(\delta)$ can be found.
Network nodes independently and randomly select linear mappings from inputs links onto outputs links over some field.

**Theorem**

In a multicast solvable network with $N$ receivers in which the coefficients for the linear combinations are chosen independently and uniformly over $\mathbb{F}_q$, the success probability that all the $N$ receivers will obtain the information sent by the source node is at least $\left(1 - \frac{N}{q}\right)^\eta$ for $q > N$, where $\eta$ is the total number of coefficients in the coding points.
Let $f(z_1, z_2, ..., z_\eta)$ be a polynomial over $\mathbb{F}_q$ such that

- $f(z_1, z_2, ..., z_\eta)$ is not identically zero;
- The degree of a variable in a term of $f$ is at most $d$;
- $q > d$.

If the values of $z_1, z_2, ..., z_\eta$ are chosen uniformly at random from $\mathbb{F}_q$ then

$$\Pr\{f(z_1, z_2, ..., z_\eta) = 0\} \leq 1 - \left(1 - \frac{d}{q}\right)^\eta$$

Weaker result

Ho, Médard, Koetter, Karger, Effros, Shi, Leong, 2006
Let $f(z_1, z_2, ..., z_\eta)$ be a polynomial over $\mathbb{F}_q$

- $f(z_1, z_2, ..., z_\eta)$ is not identically zero;
- Degree of a variable in term of $f$ is at most $d$;
- $q > d$.

If the values of $z_1, z_2, ..., z_\eta$ are chosen uniformly at random from $\mathbb{F}_q$ then

$$\Pr\{f(z_1, z_2, ..., z_\eta) = 0\} \leq 1 - \left(1 - \frac{d}{q}\right)^\eta$$

The proof is by induction on $\eta$

For $\eta = 1$, $f$ is a polynomial in a single variable of degree at most $d$.

An element of $\mathbb{F}_q$ is a root of $f$ with probability at most $\frac{d}{q} = 1 - (1 - \frac{d}{q})^1$
Random Network Coding

**Proof**

The proof is by induction on $\eta$. For $\eta = 1$, $f$ is a polynomial in a single variable of degree at most $d$. An element of $\mathbb{F}_q$ is a root of $f$ with probability at most

$$d/q = 1 - (1 - d/q)^1$$

For $\eta > 1$, we assume that the claim holds for polynomials with fewer than $\eta$ variables.

We express $f$ as

$$f(z_1, ..., z_\eta) = z_\eta^{d_1} f_1(z_1, ..., z_{\eta-1}) + f_2(z_1, ..., z_\eta),$$

where $d_1 \leq d$ is the highest degree of $z_\eta$ in $f$, $f_1$ is not identically zero, and the degrees of $z_\eta$ in $f_2$ are smaller than $d_1$.

$$\Pr(f = 0) = \Pr(f_1 = 0) \cdot \Pr(f = 0|f_1 = 0) + \Pr(f_1 \neq 0) \cdot \Pr(f = 0|f_1 \neq 0)$$
For $\eta > 1$, we assume that the claim holds for polynomials with fewer than $\eta$ variables. We express $f$ as

$$f(z_1, \ldots, z_\eta) = z_\eta^{d_1} f_1(z_1, \ldots, z_{\eta-1}) + f_2(z_1, \ldots, z_\eta),$$

where $d_1 \leq d$ and $f_1$ is not identically zero.

$$\Pr(f = 0) = \Pr(f_1 = 0) \cdot \Pr(f = 0 | f_1 = 0)$$

$$+ \Pr(f_1 \neq 0) \cdot \Pr(f = 0 | f_1 \neq 0)$$

1. $\Pr(f_1 = 0) \leq 1 - (1 - d/q)^{\eta-1}$ by the induction hypothesis

2. $\Pr(f = 0 | f_1 = 0) \leq 1$

3. $\Pr(f = 0 | f_1 \neq 0) \leq d/q$, as a polynomial in $z_\eta$
Random Network Coding

**Proof**

\[ f(z_1, \ldots, z_\eta) = z_\eta^{d_1} f_1(z_1, \ldots, z_{\eta-1}) + f_2(z_1, \ldots, z_\eta), \]

where \( d_1 \leq d \) and \( f_1 \) is not identically zero polynomial.

\[
\Pr(f = 0) = \Pr(f_1 = 0) \cdot \Pr(f = 0 | f_1 = 0) + \Pr(f_1 \neq 0) \cdot \Pr(f = 0 | f_1 \neq 0)
\]

1. \( \Pr(f_1 = 0) \leq 1 - (1 - d/q)^{\eta-1} \) by the induction hypothesis.
2. \( \Pr(f = 0 | f_1 = 0) \leq 1. \)
3. \( \Pr(f = 0 | f_1 \neq 0) \leq d/q, \) as a polynomial in \( z_\eta. \)

\[
\Pr(f = 0) \leq \Pr(f_1 = 0) + (1 - \Pr(f_1 = 0)) \frac{d}{q}
\]

by 2, 3

\[
= \Pr(f_1 = 0)\left(1 - \frac{d}{q}\right) + \frac{d}{q}
\]

\[
\leq (1 - (1 - \frac{d}{q})^{\eta-1})\left(1 - \frac{d}{q}\right) + \frac{d}{q}
\]

by 1

\[
= 1 - \left(1 - \frac{d}{q}\right)^{\eta}
\]
A multicast network is solvable if and only if the min-cut for each receiver is at least the number of messages.

A network with $h$ sources, where each one has exactly one message, is solvable if and only if the min-cut to each receiver is at least $h$. 
Random Network Coding

Multicast Networks

Network nodes independently and randomly select linear mappings from inputs links onto outputs links over some field.

Theorem

In a multicast solvable network with $N$ receivers in which the coefficients for the linear combinations are chosen independently and uniformly over $\mathbb{F}_q$, the success probability that all the $N$ receivers will obtain the information sent by the source node is at least $(1 - \frac{N}{q})^\eta$ for $q > N$, where $\eta$ is the maximum number of coding points employed by a receiver.
Let \( f(z_1, z_2, ..., z_\eta) \) be a polynomial over \( \mathbb{F}_q \) such that

- \( f(z_1, z_2, ..., z_\eta) \) is not identically zero;
- The degree of a variable in a term of \( f \) is at most \( d \); the total degree of a term is at most \( d\eta' \);
- \( q > d \).

If the values of \( z_1, z_2, ..., z_\eta \) are chosen uniformly at random from \( \mathbb{F}_q \) then

\[
\Pr\{f(z_1, z_2, ..., z_\eta) = 0\} \leq 1 - \left(1 - \frac{d}{q}\right)^{\eta'}
\]
Let $f(z_1, z_2, ..., z_\eta)$ be a nonzero polynomial over $\mathbb{F}_q$ such that the sum of degrees of all the variables in a term of $f(z_1, z_2, ..., z_\eta)$ is at most $d$. If values $a_1, a_2, ..., a_\eta \in \mathbb{F}_q$ are chosen uniformly at random from a subset $A$ of $\mathbb{F}_q$, then the probability that $f(a_1, a_2, ..., a_\eta) = 0$ is at most $d/|A|$.
The proof is by induction on $\eta$. 

For $\eta = 1$, $f$ is a polynomial in a single variable of degree at most $d$ and hence it can have at most $d$ zeros and the claim follows.

Suppose that the claim is true for all the polynomials with at most $\eta - 1$ variables, $\eta > 1$.

Let $f$ be a polynomial with $\eta$ variables, where the sum of degrees in a term is at most $d$. 
Let $f$ be a polynomial with $\eta$ variables, where the sum of degrees in a term is at most $d$. It can be written as

$$f(z_1, z_2, \ldots, z_\eta) = \sum_{i=0}^{k} z_1^i f_i(z_2, \ldots, z_\eta)$$

where $k \leq d$ is the highest degree of $z_1$ in term of $f$. $f_k(z_2, \ldots, z_\eta)$ is not identically zero and the sum of degrees of its terms is at most $d - k$. 

**Proof**
It can be written as

\[
f(z_1, z_2, \ldots, z_\eta) = \sum_{i=0}^{k} z_1^i f_i(z_2, \ldots, z_\eta)
\]

where \( k \leq d \) is the highest degree of \( z_1 \) in term of \( f \).

\( f_k(z_2, \ldots, z_\eta) \) is not identically zero and the sum of degrees of its terms is at most \( d - k \).

Hence, by the assumption the probability that \( f_k(z_2, \ldots, z_\eta) = 0 \) is at most \( \frac{d-k}{|A|} \).
Random Network Coding

**Proof**

\[ f(z_1, z_2, \ldots, z_\eta) = \sum_{i=0}^{k} z_1^i f_i(z_2, \ldots, z_\eta) \]

where \( k \leq d \) is the highest degree of \( z_1 \) in term of \( f \). \( f_k(z_2, \ldots, z_\eta) \) is not identically zero and the sum of degrees of its terms is at most \( d - k \). Hence, by the assumption the probability that \( f_k(z_2, \ldots, z_\eta) = 0 \) is at most \( \frac{d-k}{|A|} \).

If \( f_k(a_2, \ldots, a_\eta) \neq 0 \) we define

\[ g(z_1) = f(z_1, a_2, \ldots, a_\eta) = \sum_{i=0}^{k} z_1^i f_i(a_2, \ldots, a_\eta). \]

\( g(z_1) \) is a nonzero polynomial of degree \( k \) and hence the probability that \( g(a_1) = 0 \) is at most \( \frac{k}{|A|} \).
Proof

Hence, by the assumption the probability that 

\[ f_k(z_2, \ldots, z_\eta) = 0 \]

is at most \( \frac{d-k}{|A|} \). If \( f_k(a_2, \ldots, a_\eta) \neq 0 \),

\[ g(z_1) = f(z_1, a_2, \ldots, a_\eta) = \sum_{i=0}^{k} z_1^i f_i(a_2, \ldots, a_\eta). \]

\( g(z_1) \) is a nonzero polynomial of degree \( k \) and hence the probability that \( g(a_1) = 0 \) is at most \( \frac{k}{|A|} \).

Let \( B \) be the event that \( g(a_1) = f(a_1, a_2, \ldots, a_\eta) = 0 \).

Let \( C \) be the event that \( f_k(a_2, \ldots, a_\eta) = 0 \).

\[ \Pr(C) \leq \frac{d-k}{|A|}, \quad \Pr(B|\overline{C}) \leq \frac{k}{|A|} \]
Let $B$ be the event that $g(a_1) = f(a_1, a_2, \ldots, a_\eta) = 0$.

Let $C$ be the event that $f_k(a_2, \ldots, a_\eta) = 0$.

\[
\Pr(C) \leq \frac{d-k}{|A|}, \quad \Pr(B|\bar{C}) \leq \frac{k}{|A|}
\]

\[
\Pr(B) = \Pr(B|C)\Pr(C) + \Pr(B|\bar{C})\Pr(\bar{C}) \leq \Pr(C) + \Pr(B|\bar{C}) \leq \frac{d-k}{|A|} + \frac{k}{|A|} = \frac{d}{|A|}.
\]
Coherent network coding - the source and the destination nodes know the topology of the network and the network code.

Noncoherent network coding - the source and the destination nodes don’t know the topology of the network and the network code.
Introduction to Network Coding, Bounds and Constructions

END OF LECTURE 4