Introduction to Network Coding, Bounds and Constructions

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Lecture 12

Network Coding Capacity
Network Coding Capacity

Outline

Network coding capacity

NetCode capacity vs. routing capacity

Insufficiency of linear network coding
Fractional Network Code

Concepts for network coding

Messages - vectors of length $k$.
Packets on edges - vectors of length $n$.

A $(k,n)$ fractional network code is a collection of edge functions, one for each edge in the network, and demand functions, one for each demand of each node in the network.

A $(k,n)$ fractional solution is a $(k,n)$ fractional code which results in every receiver being able to deduce its demands from its input.

A network is said to be (scalar) solvable if it has a $(k,n)$ fractional solution for the case $k = n = 1$. 
A \((k, n)\) fractional coding solution is said to be **linear** if all edge functions and all demand functions are linear combination of their vector inputs, where the coefficients are matrices over the alphabet \(\Sigma\).

In a linear code, if a node has in-edges and/or source messages carrying vectors \(x_1, \ldots, x_r \in \Sigma^k \cup \Sigma^n\), then an out-edge of the node carries the vector

\[
y = \sum_{i=1}^{r} M_i x_i
\]

where each matrix \(M_i\) is over the alphabet \(\Sigma\), and is of dimension \(n \times k\) when \(x_i\) is a source message and is of dimension \(n \times n\) when \(x_i\) is an in-edge.
Coding Capacity

Concepts for network coding

The coding capacity of a network with respect to an alphabet $\Sigma$ and a class $\mathcal{C}$ of network codes is

$$\sup\left\{ \frac{k}{n} : \exists (k,n) \text{ fractional coding solution in } \mathcal{C} \text{ over } \Sigma \right\}$$

If the class $\mathcal{C}$ of network codes consists of all linear codes, then the coding capacity is referred to as the linear coding capacity, respectively.

Dougherty, Freiling, Zeger 2005
The **coding capacity** of a network is
\[
\sup \left\{ \frac{k}{n} : \exists (k, n) \text{ fractional coding solution} \right\}
\]

If a network has a \((k, n)\) fractional solution over some alphabet, then we say that the coding rate \(\frac{k}{n}\) is **achievable** for the network.
The coding capacity of any network is independent of the alphabet used.

Cannons, Dougherty, Freiling, Zeger 2006

The linear coding capacity of a network might be dependent of the alphabet used.
The coding capacity of any network is independent of the alphabet used.

**Theorem 1**
Assume a network has a \((k, n)\) fractional coding solution over an alphabet \(A\) and let \(B\) be any other alphabet of cardinality at least two.

**Proof**
Let \(\epsilon > 0\)

\[
t = \left\lfloor \frac{(k + 1) \log_2 |B|}{n\epsilon \log_2 |A|} \right\rfloor
\]

A \((tk, tn)\) solution over \(A\) is obtained by applying the \((k, n)\) solution \(t\) times.
Assume a network has a \((k, n)\) fractional coding solution over an alphabet \(A\) and let \(B\) be any other alphabet of cardinality at least two.

Proof

Let \(\epsilon > 0\)

A \((tk, tn)\) solution over \(A\) is obtained by applying the \((k, n)\) solution \(t\) times.

\[
\begin{align*}
  n' &= n \left\lceil t \frac{\log_2 |A|}{\log_2 |B|} \right\rceil \\
  k' &= \left\lfloor \frac{kn'}{n} \right\rfloor - k
\end{align*}
\]

\(|B|^{n'} \geq |A|^{tn}\) \quad |B|^{k'} \leq |A|^{tk}\)

\[
\frac{k'}{n'} \geq \frac{k}{n} - \epsilon
\]
Coding Capacity

Proof

For each edge $e$, let $d_e$ be the number of in-edges entering the starting node of $e$.

Let $m_e$ be the number of messages originating at the starting node of $e$.

For each node $v$, let $d_v$ be the number of in-edges entering the node $v$.

Let $m_v$ be the number of messages originating at $v$. 
Coding Capacity

Proof

For each edge $e$, let $f_e$ be the edge encoding function for $e$

$$f_e : (A^{tn})^{d_e} \times (A^{tk})^{m_e} \rightarrow A^{tn}$$

for each receiver node $v$ which demands the messages $m$, let $f_{v,m}$ be decoding function in $v$ for the message $m$

$$f_{v,m} : (A^{tn})^{d_v} \times (A^{tk})^{m_v} \rightarrow A^{tk}$$
**Coding Capacity**

**Proof**

- Injection: $h: A^{tn} \rightarrow B^{n'}$ any
  - $|B|^{n'} \geq |A|^{tn}$

- Such that
  - $\hat{h}(h(x)) = x, x \in A^{tn}$
  - $\hat{h}(x) = y$, any $y$, otherwise

- Injection: $h_0: B^{k'} \rightarrow A^{tk}$ any
  - $|B|^{k'} \leq |A|^{tk}$

- Such that
  - $\hat{h}_0(h_0(x)) = x, x \in B^{k'}$
  - $\hat{h}_0(x) = y$, any $y$, otherwise
For each edge $e$, define a mapping

$$g_e: (B^{n'})^{d_e} \times (B^{k'})^{m_e} \rightarrow B^{n'}$$

$$g_e(x_1, ..., x_{d_e}, y_1, ..., y_{m_e}) = h(f_e(\hat{h}(x_1), ..., \hat{h}(x_{d_e}), h_0(y_1), ..., h_0(y_{m_e})))$$

$$x_1, ..., x_{d_e} \in B^{n'}, \quad y_1, ..., y_{m_e} \in B^{k'}$$
For each receiver node $v$ which demands the message $m$, define a mapping 

$$g_{v,m}: (B^{n'})^{d_v} \times (B^{k'})^{m_v} \rightarrow B^{k'}$$

$$g_{v,m}(x_1, \ldots, x_{d_v}, y_1, \ldots, y_{m_v}) = \hat{h}_0(f_{v,m}(\hat{h}(x_1), \ldots, \hat{h}(x_{d_v}), h_0(y_1), \ldots, h_0(y_{m_v})))$$

$$x_1, \ldots, x_{d_v} \in B^{n'}, \quad y_1, \ldots, y_{m_v} \in B^{k'}$$
The \((k', n')\) fractional network code over \(B\) is obtained from the \((k', n')\) fractional network code over \(A\) by using the edge functions \(g_e\) and the decoding functions \(g_v, m\).
The \((k', n')\) fractional network code over \(B\) is obtained by using the edge functions \(g_e\) and the decoding functions \(g_{v,m}\).

The vectors on the edges in the two solutions can each be obtained from the other using \(h\) and \(\hat{h}\).

The vectors obtained by the receivers from the decoding functions in the two solutions can each be obtained from the other using \(h_0\) and \(\hat{h}_0\).
The \((k', n')\) fractional network code over \(B\) is obtained from the \((k', n')\) fractional network code over \(A\) by using the edge functions \(g_e\) and the decoding functions \(g_{v,m}\).

\[\frac{k'}{n'} \geq \frac{k}{n} - \epsilon\]

since \(\epsilon\) can be taken small as we want, it follows that the supremum of \(k/n\) and the supremum of \(k'/n'\) are equal.

The network coding capacity is the same for all alphabets.
The routing capacity of every network is rational and achievable.

There exists an algorithm for determining the routing capacity of a network.
Introduction to Network Coding, Bounds and Constructions

A SHORT BREAK
Suppose a network has a message which is demanded by a receiver $Y$ and produced by unique source $X$. If there is a unique directed path from $X$ to $Y$, then the coding capacity of the network is at most $1$. 

**Theorem 2**
Theorem 2: Suppose a network has a message $m$ which is demanded by a receiver $Y$ and produced by a unique source $X$. If there is a unique directed path from $X$ to $Y$, then the coding capacity of the network is at most 1.

Proof: Suppose there exists a $(k, n)$ fractional coding over alphabet $A$ with $n < k$

If all messages except for $m$ are fixed, then each edge of a path from $X$ to $Y$ can take on at most $|A|^n$ different values.

$Y$ can decode at most $|A|^n$ different values

$|A|^n < |A|^k$ and hence some message $m$ from $X$ cannot be decoded at $Y$, a contradiction.
The network $\mathcal{N}_1$ has a scalar linear solution over any ring with characteristic two, but has no linear solution for any vector dimension over a finite field with odd characteristic. The coding capacity of the network is 1.

The linear coding capacity of the network $\mathcal{N}_1$ is 1 over any field with even characteristic and $\frac{4}{5}$ over any field with odd characteristic.
Insufficiency of Linear Coding

Proof

\[ r(37) := a - (a - c) = c \]

\[ r(38) := (a + b - c) - (a - c) = b \]

\[ r(39) := (a + b - c) + (b + c) = a + 2b \]
### Insufficiency of Linear Coding

<table>
<thead>
<tr>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e(13, 17) = M_1a + M_2b$</td>
</tr>
<tr>
<td>$e(14, 18) = M_3b + M_4c$</td>
</tr>
<tr>
<td>$e(21, 29) = M_5e(13, 17) + M_6e(14, 18)$</td>
</tr>
<tr>
<td>$e(22, 30) = M_7e(13, 17) + M_8c$</td>
</tr>
<tr>
<td>$c = M_9a + M_{10}e(21, 29)$</td>
</tr>
<tr>
<td>$b = M_{11}e(21, 29) + M_{12}c(22, 30)$</td>
</tr>
<tr>
<td>$a = M_{13}e(22, 30) + M_{14}e(14, 18)$</td>
</tr>
</tbody>
</table>

![Diagram](image_url)
Insufficiency of Linear Coding

\[
\begin{align*}
    e(13, 17) &= M_1 a + M_2 b \\
    e(14, 18) &= M_3 b + M_4 c \\
    e(21, 29) &= M_5 e(13, 17) + M_6 e(14, 18) \\
    e(22, 30) &= M_7 e(13, 17) + M_8 c \\
    c &= M_9 a + M_{10} e(21, 29) \\
    b &= M_{11} e(21, 29) + M_{12} c(22, 30) \\
    a &= M_{13} e(22, 30) + M_{14} e(14, 18)
\end{align*}
\]

\[
\begin{align*}
    c &= (M_9 + M_{10} M_5 M_1) a + (M_{10} M_5 M_2 + M_{10} M_6 M_3) b + M_{10} M_6 M_4 c \\
    b &= (M_{11} M_5 M_1 + M_{12} M_7 M_1) a + (M_{11} M_6 M_4 + M_{12} M_8) c \\
    &\quad + (M_{11} M_5 M_2 + M_{11} M_6 M_3 + M_{12} M_7 M_2) b \\
    a &= (M_{13} M_7 M_2 + M_{14} M_3) b + (M_{13} M_8 + M_{14} M_4) c + M_{13} M_7 M_1 a
\end{align*}
\]
Insufficiency of Linear Coding

\[ c = (M_9 + M_{10}M_5M_1)a + (M_{10}M_5M_2 + M_{10}M_6M_3)b + M_{10}M_6M_4c \]

\[ b = (M_{11}M_5M_1 + M_{12}M_7M_1)a + (M_{11}M_6M_4 + M_{12}M_8)c + (M_{11}M_5M_2 + M_{11}M_6M_3 + M_{12}M_7M_2)b \]

\[ a = (M_{13}M_7M_2 + M_{14}M_3)b + (M_{13}M_8 + M_{14}M_4)c + M_{13}M_7M_1a \]

\[ M_9 + M_{10}M_5M_1 = 0 \]
\[ M_{10}(M_5M_2 + M_6M_3) = 0 \]
\[ M_{10}M_6M_4 = I \]

\[ M_{13}M_7M_2 + M_{14}M_3 = 0 \]
\[ M_{13}M_8 + M_{14}M_4 = 0 \]
\[ M_{13}M_7M_1 = I \]

\[ M_{11}M_5M_1 + M_{12}M_7M_1 = 0 \]
\[ M_{11}M_6M_4 + M_{12}M_8 = 0 \]
\[ M_{11}M_5M_2 + M_{11}M_6M_3 + M_{12}M_7M_2 = I \]
Insufficiency of Linear Coding

\[ M_9 + M_{10}M_5M_1 = 0 \]
\[ M_{10}(M_5M_2 + M_6M_3) = 0 \]
\[ M_{10}M_6M_4 = I \]

\[ M_{13}M_7M_2 + M_{14}M_3 = 0 \]
\[ M_{13}M_8 + M_{14}M_4 = 0 \]
\[ M_{13}M_7M_1 = I \]

\[ M_{11}M_5M_1 + M_{12}M_7M_1 = 0 \]
\[ M_{11}M_6M_4 + M_{12}M_8 = 0 \]
\[ M_{11}M_5M_2 + M_{11}M_6M_3 + M_{12}M_7M_2 = I \]

\[ M_1, M_4, M_6, M_7, M_{10}, M_{13} \text{ are invertible.} \]

\[ M_{10}(M_5M_2 + M_6M_3) = 0 \]
\[ \Rightarrow M_5M_2 + M_6M_3 = 0 \]

\[ M_{11}M_5M_2 + M_{11}M_6M_3 + M_{12}M_7M_2 = I \]
\[ \Rightarrow M_{12}M_7M_2 = I \]

\[ M_1, M_4, M_6, M_7, M_{10}, M_{13}, M_2, M_{12} \text{ are invertible.} \]
Insufficiency of Linear Coding

\[ M_9 + M_{10}M_5M_1 = 0 \]
\[ M_{10}(M_5M_2 + M_6M_3) = 0 \]
\[ M_{10}M_6M_4 = I \]

\[ M_{13}M_7M_2 + M_{14}M_3 = 0 \]
\[ M_{13}M_8 + M_{14}M_4 = 0 \]
\[ M_{13}M_7M_1 = I \]

\[ M_{11}M_5M_1 + M_{12}M_7M_1 = 0 \]
\[ M_{11}M_6M_4 + M_{12}M_8 = 0 \]
\[ M_{11}M_5M_2 + M_{11}M_6M_3 + M_{12}M_7M_2 = I \]

\( M_1, M_4, M_6, M_7, M_{10}, M_{13}, M_2, M_{12} \) are invertible.

\[ M_{11}M_5M_1 + M_{12}M_7M_1 = 0 \]
\[ M_{13}M_7M_2 + M_{14}M_3 = 0 \]

\( M_1, M_4, M_6, M_7, M_{10}, M_{13}, M_2, M_{12}, M_5, M_{11}, M_3, M_{14} \) are invertible.
Insufficiency of Linear Coding

\[ M_9 + M_{10}M_5M_1 = 0 \]
\[ M_{10}(M_5M_2 + M_6M_3) = 0 \]
\[ M_{10}M_6M_4 = I \]

\[ M_{13}M_7M_2 + M_{14}M_3 = 0 \]
\[ M_{13}M_8 + M_{14}M_4 = 0 \]
\[ M_{13}M_7M_1 = I \]

\[ M_{11}M_5M_1 + M_{12}M_7M_1 = 0 \]
\[ M_{11}M_6M_4 + M_{12}M_8 = 0 \]
\[ M_{11}M_5M_2 + M_{11}M_6M_3 + M_{12}M_7M_2 = I \]

\[ M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_{10}, M_{11}, M_{12}, M_{13}, M_{14} \] are invertible.

\[ M_9 + M_{10}M_5M_1 = 0 \]
\[ M_{13}M_8 + M_{14}M_4 = 0 \]

\[ M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_{10}, M_{11}, M_{12}, M_{13}, M_{14}, M_8, M_9 \] invertible.

⇒ all invertible.
Insufficiency of Linear Coding

\[ M_{11}M_5M_1 + M_{12}M_7M_1 = 0 \]

\[ \Rightarrow M_{12}M_7M_2 = I \]

\[ M_{11}M_6M_4 + M_{12}M_8 = 0 \]

\[ \Rightarrow -I = M_{11}M_5M_2 \]

\[ M_{13}M_8 + M_{14}M_4 = 0 \]

\[ 0 = M_{11}M_6M_4 + M_{12}M_8 \]

\[ = M_{11}(M_6M_3)M_3^{-1}M_4 \]

\[ - M_{12}M_3^{-1}M_{14}M_4 \]

\[ = -M_{11}M_5M_2M_3^{-1}M_4 \]

\[ + (M_{12}M_7M_2)M_3^{-1}M_4 \]

\[ = (-M_{11}M_5M_2 + I)M_3^{-1}M_4 \]

\[ \Rightarrow I = M_{11}M_5M_2 \]
Insufficiency of Linear Coding

The network $\mathcal{N}_1$ has a scalar linear solution over $\text{GF}(2)$, and hence its coding capacity (which is independent of alphabet size by Theorem 1) is at least 1. Since there is a unique path from node 6 to node 37, it follows by Theorem 2 that the coding capacity is at most 1. Thus, the network coding capacity is exactly 1.
Insufficiency of Linear Coding

Lemma

The network $\mathcal{N}_3$ has a scalar linear solution over any ring where $2$ is a unit, but has no linear solution for any vector dimension over a finite field with characteristic two. The coding capacity of the network is 1.

The linear coding capacity of the network $\mathcal{N}_3$ is 1 over any field with odd characteristic and $\frac{10}{11}$ over any field with even characteristic.
Insufficiency of Linear Coding

\[ r(40) := (a + b + c) - (a + b) = c \]

\[ r(41) := (a + b + c) - (a + c) = b \]

\[ r(42) := (a + b + c) - (b + c) = a \]

\[ r(43) := 2^{-1}((a + c) + (b + c) - (a + b)) = c \]

\[ r(44) := (c + d + e) - (c + d) = e \]

\[ r(45) := (c + d + e) - (c + e) = d \]

\[ r(46) := (c + d + e) - (d + e) = c \]
Insufficiency of Linear Coding

**Theorem**

There exists a solvable network that has no linear solution over any finite field and any vector dimension. The capacity of the network is $1$. 

$N_4$
Insufficiency of Linear Coding

\[ \mathcal{N}_1 \]

\[ \mathcal{N}_3 \]

\[ \mathcal{N}_4 \]
Insufficiency of Linear Coding

$N_4$
Solution over an alphabet of size 4. We use both $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. The operations used are $+$ and $-$ for addition and subtraction in $\mathbb{Z}_4$, $\oplus$ addition modulo 2, and $t(x, y) = (y, x)$. 
Insufficiency of Linear Coding

\[ r(37) := a \oplus (a \oplus c) = c \]

\[ r(38) := (a \oplus c) \oplus (a \oplus b \oplus c) = b \]

\[ r(39) := (a \oplus b \oplus c) \oplus (b \oplus c) = a \]

\[ r(40) := (a + b + c) - (a + b) = c \]

\[ r(41) := (a + b + c) - (a + c) = b \]

\[ r(42) := (a + b + c) - (b + c) = a \]

\[ r(44) := (t(c) + d + e) - (t(c) + d) = e \]

\[ r(45) := (t(c) + d + e) - (t(c) + e) = d \]

\[ r(46) := t((t(c) + d + e) - (d + e)) = c \]
Insufficiency of Linear Coding

0 = (0, 0), 1 = (0, 1)
2 = (1, 0), 3 = (1, 1)

c = (x, y)
2c = (y, 0)
t(c) = (y, x)
t(2c) = (0, y)
2t(c) = (x, 0)

2t(c) + t(2c) = (x, y) = c

\[ r(43) := t((a + c) + (b + c) - (a + b)) + (t(c) + d) + (t(c) + e) - (d + e) = t(2c) + 2t(c) \]
Insufficiency of Linear Coding

The network $\mathcal{N}_4$ is solvable and hence its coding capacity (which is independent of alphabet size by Theorem 1) is at least 1. Since there is a unique path from node 6 to node 37, it follows by Theorem 2 that the coding capacity is at most 1. Thus, the network coding capacity is exactly 1.
Insufficiency of Linear Coding

Corollary

The linear coding capacity of the network $\mathcal{N}_4$ is $\frac{4}{5}$ over any field with odd characteristic and $\frac{10}{11}$ over any field with even characteristic.
A class of network codes is **asymptotically sufficient** over a class of alphabets if every solvable network is asymptotically solvable in the class of codes over some member of the alphabet class.

A network is **asymptotically solvable** with respect to an alphabet and a class of codes if its coding capacity is at least 1.

Class of network codes is **sufficient** over a class of alphabets if every solvable network has a solution in the class of codes over some member of the alphabet class.
**Corollary**

The linear coding capacity of the network $\mathcal{N}_4$ is $\frac{4}{5}$ over any field with odd characteristic and $\frac{10}{11}$ over any field with even characteristic.

**Lemma**

The coding capacity of the network $\mathcal{N}_4$ is 1.

**Corollary**

There exists a solvable network which is not asymptotically linearly solvable. Linear network codes are asymptotically insufficient over finite fields.
A SHORT BREAK
Two networks are disjoint if their node sets are disjoint.

A network is said to be the union of two disjoint networks if its underlying graph is the union of the underlying graphs of the two disjoint networks (no edges between the two subnetworks), and the assignments of messages and demands to the nodes remain unchanged.
Unachievability of Coding Capacity

**Lemma 1**

Every rational coding rate less than the coding capacity is achievable.

**Dougherty, Freiling, Zeger 2006**

**Proof**

Let $\frac{k}{n}$ be a rational number less than the network capacity.

There exists a $(k', n')$ fractional coding solution over some alphabet such that $\frac{k}{n} < \frac{k'}{n'}$. 

$(k', n')$ solution implies $(kk', kn')$ solution

increased edge dimension implies $(kk', nk')$ solution

rate $\frac{k}{n}$ achieved
Unachievability of Coding Capacity

Theorem 3

Dougherty, Freiling, Zeger 2006

If two disjoint solvable networks are never solvable over the same alphabet, then their union has coding capacity equal to 1, which cannot be achieved.

Proof

Suppose two disjoint networks $N_1$ and $N_2$ are solvable with alphabets $A$ and $B$, respectively, where $|A| \neq |B|$, but are never solved over the same alphabet.

Solvability implies that the coding capacities of $N_1$ and $N_2$ are at least 1 over the alphabets $A$ and $B$, respectively.
Unachievability of Coding Capacity

Proof
Suppose two disjoint networks $N_1$ and $N_2$ are solvable with alphabets $A$ and $B$, respectively, where $|A| \neq |B|$, but are never solved over the same alphabet.

Solvability implies that the coding capacities of $N_1$ and $N_2$ are at least 1 over the alphabets $A$ and $B$, respectively.

By Theorem 1, the network capacities of the two networks is 1 over any alphabet.

For any $\epsilon > 0$ there exists a $(k, n)$ solution over the alphabet $A$ to the network $N_2$ such that $k/n > 1 - \epsilon$.

If $k > n$ then the $(k, n)$ solution implies a $(k, \max\{k, n\})$ solution.

$N_1$ has a $(1, 1)$ and a $(k, k)$ solutions over $A$. 

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Unachievability of Coding Capacity

Proof: By Theorem 1, the network capacities of the two networks is 1 over any alphabet.

For any $\epsilon > 0$ there exists a $(k, n)$ solution over the alphabet $A$ to the network $N_2$ such that $\frac{k}{n} > 1 - \epsilon$.

If $k > n$ then the $(k, n)$ solution implies a $(k, \max\{k, n\})$ solution.

If $n > k$ then the $(k, k)$ solution implies a $(k, n)$ solution over $A$.

Hence, the union network $N_1 \cup N_2$ has a $(k, \max\{k, n\})$ solution over $A$.

$k / \max\{k, n\} > 1 - \epsilon$ implies that $
\sup_{\epsilon > 0} \frac{k}{\max\{k, n\}} \geq 1$.

Network capacity of $N_1 \cup N_2$ is at least 1.
Unachievability of Coding Capacity

Proof

Network capacity of $N_1 \cup N_2$ is at least 1

Suppose there exists $(k, n)$ fractional coding solution for $N_1 \cup N_2$ over some alphabet $D$ with $k \geq n$

For each of the two networks this solution induces a $(k, k)$ fractional coding solution over $D$ by leaving $k - n$ edge components unused

This induces scalar solutions for $N_1$ and $N_2$ over an alphabet of size $|D|^k$
Proof

This induces scalar solutions for \( N_1 \) and \( N_2 \) over an alphabet of size \( |D|^k \).

Contradiction to the assumption that the two network are not solvable over the same alphabet.

Any \((k, n)\) fractional coding solution must have \( k < n \) and hence 1 (which is at least the coding capacity of the network) is not an achievable coding rate of \( N_1 \cup N_2 \).

Hence the coding capacity of \( N_1 \cup N_2 \) is 1 and this coding capacity is not achievable.
Unachievability of Coding Capacity

**Lemma** The network $\mathcal{N}_1$ has a scalar linear solution over any ring with characteristic two, but has no linear solution for any vector dimension over a finite field with odd characteristic. The coding capacity of the network is 1.

**Theorem 4** The network $\mathcal{N}_1$ is solvable if and only if the alphabet size is a power of 2.

**Dougherty, Freiling, Zeger 2006**

Using equations and entropy inequalities.
Unachievability of Coding Capacity

Theorem 5

The network $\mathcal{N}_2$ is solvable if and only if the alphabet size is odd.

Dougherty, Freiling, Zeger 2006
Unachievability of Coding Capacity

Theorem

There exists a directed acyclic network whose coding capacity is not achievable.

Dougherty, Freiling, Zeger 2006

Proof

Follows immediately from Theorem 3, Theorems 4, and Theorem 5 related to the two disjoint networks $\mathcal{N}_1$ and $\mathcal{N}_2$. 
Research Problems

Given a routing capacity of a network, what is the smallest $k$ for which this capacity is attained?

Given a coding capacity of a network, what is the smallest $k$ for which this capacity is attained (if there exists such a $k$)?
Introduction to Network Coding, Bounds and Constructions

END OF LECTURE 12