Exercise 1
Consider the function $f(x) = \frac{\sin x}{x^3}$.

(a) Using $f(4), f(4-h), f(4+h)$ approximate $f'(x=4)$.
(b) Using $f(4), f(4+h)$ approximate $f'(x=4)$.
(c) Using $f(4), f(4+h), f(4+\frac{3}{2}h)$ approximate $f'(x=4)$.
(d) Using $f(4), f(4-h), f(4+h)$ approximate $f''(x=4)$.

For each (a)-(d) find the analytical expression of the approximation; calculate the numerical error $R_T$ and plot the graph of $\log(R_T)$ as a function of $\log(h)$ (here $h=\Delta x$) for $10^{-4} \leq h \leq 0.2$ with single (of 6 digits) and double (of 12 digits) precisions.

Exercise 2
Consider the problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, & 0 < x < 1, & t > 0; \\
u(x,0) &= \begin{cases} 
2x, & 0 \leq x \leq 0.5 \\
-x+2, & 0.5 \leq x \leq 1 
\end{cases} \\
u(0,t) &= 0, & t \geq 0; \\
u(1,t) &= 1, & t \geq 0;
\end{align*}
\]

(a) Solve the problem numerically by explicit scheme using various values of $h$: $h=0.1, h=0.05, h=0.01$ and $k=10^{-3}$.

(b) Solve the problem numerically by implicit Crank-Nikolson scheme using $h = 0.1, \quad r = 0.6$.

(c) Plot the solutions $u(x,t)$ obtained by (a)-(b) at $t=0.05, 0.10, 0.20, 0.30$. 
(d) Find the analytical solution of the problem and compare it with the numerical solutions from the previous sections.

**Hints to section (d):**
1). The boundary conditions of the given problem are non-homogeneous. Assume \( u(x,t) = w(x,t) + x \), submit it into the problem and derive a new formulation of the problem for \( w(x,t) \).
2). Use Fourier series to develop the initial condition function into the series of \( \sin \): for example for \( w(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin(\pi nx) \), the coefficients will be given by

\[
a_n = 2 \int_{0}^{1} f(x) \sin(\pi nx) \, dx, \quad n = 1, 2, \ldots.
\]