Exercise 1.
Consider the equation
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1
\]
\[
u(0, t) = U_0(t), \quad t \geq 0
\]
\[
u(1, t) = U_n(t), \quad t \geq 0
\]
\[
u(x, 0) = F(x)
\]
Let us define \( t_j = jk, \quad j = 0, 1, \ldots \)
For \( x \) non-equal intervals grid is defined:
\[
x_i = x_{i-1} + h_i, \quad 0 \leq i \leq N
\]
\[
x_0 = 0,
\]
\[
h_i = Hq^i, 0 < q \leq 1
\]
where \( \sum_{i=1}^{N} h_i = 1 \) (this condition defines the value of \( H \)).

q and N are known constants.
We suggest the scheme:
\[
u_{i,j+1} = \nu_{i,j} + \frac{k}{H^2} [\alpha_{i-1} \nu_{i-1,j} - \beta_i \nu_{i,j} + \alpha_{i+1} \nu_{i+1,j}]
\]
a). under the assumption that the finite-difference approximation of \( \frac{\partial^2 u}{\partial x^2} \) is as better as possible, prove that \( \beta_i = \alpha_{i-1} + \alpha_{i+1} \)
b). calculate the values of \( \alpha_{i-1}, \alpha_{i+1} \)

c). Using the previous sections, check the stability conditions of the scheme in terms of \( \alpha, H, k \) and \( N \) (you can choose any method for it). Is the result expected? According your results, what can you say about the convergence of the scheme?

Exercise 2
Consider
\[
\frac{\partial u}{\partial t} + \pi \frac{\partial u}{\partial x} = e^{-t}, \quad u = u(x, t).
\]

It is known that the truncation error for the finite-differences scheme with constant intervals \( \delta x, \delta t \) is given by
\[
T_{i,j} = \left( \frac{\partial u}{\partial t} + \pi \frac{\partial u}{\partial x} - e^{-t} \right)_{i,j} + \frac{\delta t}{2} u_{i+1} - c \frac{\delta x}{2} u_{i+1} + \frac{c(\delta x)^2}{6} \left[ 2c^2 \left( \frac{\delta t}{\delta x} \right)^2 - 3c \left( \frac{\delta t}{\delta x} \right) + 1 \right] u_{xxx},
\]
where \( c = \pi \).

a). What is the necessary condition for the convergence?

b). Is the scheme consistence? Show which values of \( r \) were checked.

c). The initial conditions of the problem are:

\[ u(t = 0, x) = \sin x, \quad x \geq 0. \]

Find the analytical solution \( u(\frac{\pi}{2}, \frac{\pi}{2}) \).

d). What is the behavior of dispersion/dissipation? (explain shortly the terms).

Assume \( \delta x = 0.01 \). Which value of \( \delta t \) you would choose to minimize the dissipation of the problem?

Good luck!