Question #1 (15 points) – Uniform Sampling

Consider the signal:

\[ \varphi(t) = \begin{cases} 
4t & \text{for } t \in \left[0, \frac{1}{4}\right) \\
1 & \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right) \\
1 - 4\left(t - \frac{3}{4}\right) & \text{for } t \in \left[\frac{3}{4}, 1\right) 
\end{cases} \]

that is graphically described as

The signal is uniformly sampled based on \( N \) equal-length intervals, resulting in a piecewise-constant reconstruction of the form

\[ \hat{\varphi}_N(t) = \varphi_i \quad \text{for } t \in \left[\frac{i-1}{N}, \frac{i}{N}\right) \]

where \( i = 1, \ldots, N \) denotes the sampling interval index, and \( \varphi_i \) is the corresponding sample (a real scalar).

The corresponding Mean Squared Error is defined as

\[ MSE_N = \int_{t=0}^{1} (\varphi(t) - \hat{\varphi}_N(t))^2 \, dt \]
a. For \( N = 4 \), what are the samples \( \varphi_1, \ldots, \varphi_N \) and the corresponding \( MSE_N \)?

**Solution:**

The four sampling intervals for uniform sampling are \([0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}), [\frac{3}{4}, 1)\).

The optimal samples are the signal averages in the respective sampling intervals (we use here the fact that the average of a linear function is its value in its center):

\( \varphi_1 = \frac{1}{2} \), \( \varphi_2 = 1 \), \( \varphi_3 = 1 \), \( \varphi_4 = \frac{1}{2} \)

and the corresponding MSE is

\[
MSE_{N=4} = \int_{t=0}^{\frac{1}{4}} (\varphi(t) - \frac{1}{2})^2 \, dt + \int_{t=\frac{1}{4}}^{\frac{1}{2}} (\varphi(t) - 1)^2 \, dt + \int_{t=\frac{1}{2}}^{\frac{3}{4}} (\varphi(t) - 1)^2 \, dt + \int_{t=\frac{3}{4}}^{1} (\varphi(t) - \frac{1}{2})^2 \, dt = 
\]

\[
= \int_{t=0}^{\frac{1}{4}} (4t - \frac{1}{2})^2 \, dt + \int_{t=\frac{1}{4}}^{\frac{1}{2}} \frac{3}{4} \, dt + \int_{t=\frac{1}{2}}^{\frac{3}{4}} \frac{3}{4} \, dt + \int_{t=\frac{3}{4}}^{1} \frac{1}{4} \, dt
\]

where the last equality is due to

\[
\int_{t=\frac{3}{4}}^{1} \left( \frac{1}{2} - 4 \left( t - \frac{3}{4} \right) \right)^2 \, dt = \int_{t=\frac{3}{4}}^{1} \left( \frac{1}{2} + 4 \left( t - \frac{3}{4} \right) \right)^2 \, dt = \int_{t=0}^{\frac{1}{4}} \left( \frac{1}{2} + 4 \bar{\ell} \right)^2 \, d\bar{\ell}
\]

where we applied integration variable substitution of \( \bar{\ell} = t - \frac{3}{4} \).

Returning to the MSE:

\[
MSE_{N=4} = \frac{1}{4} \int_{t=0}^{\frac{1}{4}} (4t - \frac{1}{2})^2 \, dt = \frac{1}{4} \cdot \frac{1}{3} \left( 4t - \frac{1}{2} \right)^2 \bigg|_{t=0}^{\frac{1}{4}} = \frac{1}{6} \left( 1 - \frac{1}{2} \right)^3 - \frac{1}{6} \left( -\frac{1}{2} \right)^3 = \frac{1}{24}
\]

b. For \( N = 8 \), what are the samples \( \varphi_1, \ldots, \varphi_N \) and the corresponding \( MSE_N \)?

**Solution:**

The optimal samples are the signal averages in the sampling intervals here (we use here the fact that the average of a linear function is its value in its center):

\( \varphi_1 = \frac{1}{4} \), \( \varphi_2 = \frac{3}{4} \), \( \varphi_3 = 1 \), \( \varphi_4 = 1 \), \( \varphi_5 = 1 \), \( \varphi_6 = 1 \), \( \varphi_7 = \frac{3}{4} \), \( \varphi_8 = \frac{1}{4} \)
\[ MSE_{N=8} = \int_{t=0}^{\frac{1}{8}} (\varphi(t) - \frac{1}{4})^2 \, dt + \int_{t=\frac{1}{4}}^{\frac{3}{4}} (\varphi(t) - \frac{3}{4})^2 \, dt + \int_{t=\frac{3}{4}}^{\frac{7}{8}} (\varphi(t) - 1)^2 \, dt + \int_{t=\frac{7}{8}}^{1} (\varphi(t) - \frac{3}{4})^2 \, dt \]

\[ + \int_{t=\frac{7}{8}}^{1} (\varphi(t) - \frac{1}{4})^2 \, dt = \]

\[ = 2 \int_{t=0}^{\frac{1}{8}} (\varphi(t) - \frac{1}{4})^2 \, dt + 2 \int_{t=\frac{1}{4}}^{\frac{3}{4}} (\varphi(t) - \frac{3}{4})^2 \, dt = \]

\[ = 2 \int_{t=0}^{\frac{1}{8}} (4t - \frac{1}{4})^2 \, dt + 2 \int_{t=\frac{1}{4}}^{\frac{3}{4}} (4t - \frac{3}{4})^2 \, dt = \]

\[ = 4 \int_{t=0}^{\frac{1}{8}} (4t - \frac{1}{4})^2 \, dt \]

\[ = 4 \cdot \frac{1}{4} \cdot \frac{1}{3} (4t - \frac{1}{4}) \bigg|_{t=0}^{\frac{1}{8}} = \frac{1}{96} \]

Note: in the above solution appear direct calculations of the optimal MSE values. One can also rely on specific formulas developed in class, however, applying the required updates here.
Question #2 (25 points) –

Signal Discretization using a Piecewise-Linear Approximation

In class we discussed a signal sampling procedure that relies on a piecewise-constant approximation of the given signal. In this question, we extend the sampling procedure to rely on a piecewise-linear approximation of the signal.

The given signal, \( \varphi(t) \), is defined for \( t \in [0,1) \) as a mapping to the range of values \([\varphi_L, \varphi_H]\).

Let us consider a discretization procedure based on a uniform segmentation of the unit interval into \( N \) intervals of equal size, i.e.,

\[
\Delta_i = \left[ \frac{i-1}{N}, \frac{i}{N} \right], \quad i = 1, \ldots, N.
\]

The approximated signal, \( \hat{\varphi}(t) \), is formed from linear approximations, each associated with an interval:

For \( t \in \left[ \frac{i-1}{N}, \frac{i}{N} \right] \):

\[
\hat{\varphi}(t) = a_i(t - t_i) + c_i
\]

where \( a_i \) and \( c_i \) are real-valued scalar constants defining the linear approximation of the \( i^{th} \) interval, and \( t_i \) is the center of the \( i^{th} \) interval.

The approximations are evaluated here for the MSE criterion.

a. Show that for a positive integer \( k \):

\[
\int_{t \in \Delta_i} (t - t_i)^k \, dt = \begin{cases} 
0, & k \text{ is odd} \\
\frac{|\Delta_i|^{k+1}}{2^k \cdot (k+1)}, & k \text{ is even}
\end{cases}
\]

where \( |\Delta_i| \) is the size of the interval.

Solution:

\[
\int_{t \in \Delta_i} (t - t_i)^k \, dt = \int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} (t - t_i)^k \, dt = \frac{(t - t_i)^{k+1}}{k+1} \bigg|_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} = \frac{\left( \frac{|\Delta_i|}{2} \right)^{k+1} - \left( -\frac{|\Delta_i|}{2} \right)^{k+1}}{k+1}
\]

\[
= \frac{1 + (-1)^k |\Delta_i|^{k+1}}{2^{k+1} \cdot (k+1)} = \begin{cases} 
0, & k \text{ is odd} \\
\frac{|\Delta_i|^{k+1}}{2^k \cdot (k+1)}, & k \text{ is even}
\end{cases}
\]
b. What are the **optimal coefficients** \( a_i \) and \( c_i \) that minimize the MSE of representing the entire signal using \( N \) intervals?

**Solution:**
The MSE of the discussed representation has the form of

\[
MSE_{lin} = \int_0^1 (\varphi(t) - \varphi(t))^2 dt = \sum_{i=1}^{N} \int_{t_{i-\frac{\Delta_i}{2}}}^{t_{i+\frac{\Delta_i}{2}}} (\varphi(t) - a_i(t - t_i) - c_i)^2 dt
\]

Optimizing the representation constants \( a_i \) and \( c_i \) \((i = 1, \ldots, N)\) is done by

\[
\frac{\partial}{\partial a_i} MSE_{lin} = 0 \\
\frac{\partial}{\partial c_i} MSE_{lin} = 0
\]

Let's start with optimizing \( c_i \):

\[
\frac{\partial}{\partial c_i} MSE_{lin} = \frac{t_{i-\frac{\Delta_i}{2}}}{t_{i+\frac{\Delta_i}{2}}} \int_{t_{i-\frac{\Delta_i}{2}}}^{t_{i+\frac{\Delta_i}{2}}} (\varphi(t) - a_i(t - t_i) - c_i)^2 dt = -2 \int_{t_{i-\frac{\Delta_i}{2}}}^{t_{i+\frac{\Delta_i}{2}}} (\varphi(t) - a_i(t - t_i) - c_i) dt
\]

Based on subsection (a):

\[
\int_{t_{i-\frac{\Delta_i}{2}}}^{t_{i+\frac{\Delta_i}{2}}} (t - t_i) dt = 0
\]

and

\[
\int_{t_{i-\frac{\Delta_i}{2}}}^{t_{i+\frac{\Delta_i}{2}}} dt = |\Delta_i|
\]

Hence
\[
\frac{\partial}{\partial c_i} \text{MSE}_{i\text{lin}} = -2 \int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} \varphi(t) dt + 2c_i |\Delta_i|
\]

and the optimality equation \(\frac{\partial}{\partial c_i} \text{MSE}_{i\text{lin}} = 0\) leads to

\[
c_i^{opt} = \frac{1}{|\Delta_i|} \int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} \varphi(t) dt
\]

Now we will optimize \(a_i\):

\[
\frac{\partial}{\partial a_i} \text{MSE}_{i\text{lin}} = \frac{\partial}{\partial a_i} \left( \int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} (\varphi(t) - a_i(t - t_i) - c_i)^2 dt \right) = -2 \int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} (t - t_i) (\varphi(t) - a_i(t - t_i) - c_i) dt =
\]

\[
= -2 \int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} \varphi(t)(t - t_i) dt + 2a_i \int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} (t - t_i)^2 dt + 2c_i \int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} (t - t_i) dt
\]

Based on subsection (a):

\[
\int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} (t - t_i) dt = 0
\]

and

\[
\int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} (t - t_i)^2 dt = \frac{|\Delta_i|^3}{12}
\]

Hence

\[
\frac{\partial}{\partial a_i} \text{MSE}_{i\text{lin}} = -2 \int_{t_i - \frac{|\Delta_i|}{2}}^{t_i + \frac{|\Delta_i|}{2}} \varphi(t)(t - t_i) dt + 2a_i \frac{|\Delta_i|^3}{12}
\]

and the optimality equation \(\frac{\partial}{\partial a_i} \text{MSE}_{i\text{lin}} = 0\) leads to
\[ a_i^{\text{opt}} = \frac{12}{|\Delta|} \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} \phi(t)(t - t_i)dt \]

c. Formulate the minimal MSE of representing the entire signal using \( N \) intervals.

**Solution:**
The minimal MSE is obtained by using the above \( a_i^{\text{opt}} \) and \( c_i^{\text{opt}} \) to form \( \hat{\phi}^{\text{opt}}(t) \). Then,

\[
MSE_{\text{lin}}^{\text{opt}} = \frac{1}{0} (\phi(t) - \hat{\phi}^{\text{opt}}(t))^2 dt = \sum_{i=1}^{N} \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} (\phi(t) - a_i^{\text{opt}}(t - t_i) - c_i^{\text{opt}})^2 dt
\]

\[
= \sum_{i=1}^{N} \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} \left( \phi^2(t) + (a_i^{\text{opt}})^2(t - t_i)^2 + (c_i^{\text{opt}})^2 - 2a_i^{\text{opt}}(t - t_i)\phi(t) - 2c_i^{\text{opt}}\phi(t) \right)
+ 2a_i^{\text{opt}}c_i^{\text{opt}}(t - t_i) dt
\]

\[
= \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} \phi^2(t) dt + \sum_{i=1}^{N} \left( a_i^{\text{opt}} \right)^2 \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} (t - t_i)^2 dt + |\Delta| \sum_{i=1}^{N} \left( c_i^{\text{opt}} \right)^2
- 2 \sum_{i=1}^{N} a_i^{\text{opt}} \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} \phi(t)(t - t_i) dt - 2 \sum_{i=1}^{N} c_i^{\text{opt}} \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} \phi(t) dt
+ 2 \sum_{i=1}^{N} a_i^{\text{opt}} c_i^{\text{opt}} \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} (t - t_i) dt
\]

Using \( \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} (t - t_i) dt = 0 \) and \( \int_{t_i-|\Delta|/2}^{t_i+|\Delta|/2} (t - t_i)^2 dt = \frac{|\Delta|^3}{12} \) and the formulations for \( a_i^{\text{opt}} \) and \( c_i^{\text{opt}} \) gives (recall also that all the intervals are equally size \(|\Delta| = |\Delta|\))

\[
MSE_{\text{lin}}^{\text{opt}} = \int_{0}^{1} \phi^2(t) dt + \frac{|\Delta|^3}{12} \sum_{i=1}^{N} \left( a_i^{\text{opt}} \right)^2 + |\Delta| \sum_{i=1}^{N} \left( c_i^{\text{opt}} \right)^2
- 2 \frac{|\Delta|^3}{12} \sum_{i=1}^{N} \left( a_i^{\text{opt}} \right)^2 - 2|\Delta| \sum_{i=1}^{N} \left( c_i^{\text{opt}} \right)^2
\]

\[
= \int_{0}^{1} \phi^2(t) dt - \frac{|\Delta|^3}{12} \sum_{i=1}^{N} \left( a_i^{\text{opt}} \right)^2 - |\Delta| \sum_{i=1}^{N} \left( c_i^{\text{opt}} \right)^2
\]
\[ \int_0^1 \varphi^2(t) \, dt - \frac{1}{12N^3} \sum_{i=1}^{N} (a_i^{opt})^2 - \frac{1}{N} \sum_{i=1}^{N} (c_i^{opt})^2 \]

d. Compare the minimal MSE for using **piecewise-linear** approximation and the minimal MSE for using **piecewise-constant** approximation (as given in class – no need to develop it).

**Which MSE is lower?** Mathematically justify your answer.

**Solution:**
We saw in class that the minimal MSE for a piecewise-constant approximation of a signal is

\[ \text{MSE}_{\text{const}}^{opt} = \int_0^1 \varphi^2(t) \, dt - \frac{1}{N} \sum_{i=1}^{N} (c_i^{opt})^2 \]

Therefore, by the expression for the minimal MSE of piecewise-linear approximation we get that

\[ \text{MSE}_{\text{lin}}^{opt} - \text{MSE}_{\text{const}}^{opt} = -\frac{1}{12N^3} \sum_{i=1}^{N} (a_i^{opt})^2 \]

and since

\[ \frac{1}{12N^3} \sum_{i=1}^{N} (a_i^{opt})^2 \geq 0 \]

We conclude that

\[ \text{MSE}_{\text{lin}}^{opt} \leq \text{MSE}_{\text{const}}^{opt} \]

Showing that the extended model of piecewise-linear form improves the approximation obtained using piecewise-constant approximation.

The MSE values will be equal when the input signal has a piecewise-constant form with respect to the uniform segmentation of \([0,1)\) to \(N\) equally sized intervals.
**Question #3 (20 points) – Bit Allocation of a One-Dimensional Signal**

Consider the following signal for $t \in [0,1)$:

$$\varphi(t) = A \cdot \sin(2\pi \omega t + \phi)$$

where $A$, $\omega$, and $\phi$ are the signal’s amplitude, frequency and phase parameters, respectively. We assume here that $\omega$ is an integer.

We would like to find the optimal bit-allocation for $\varphi(t)$.

a. Express the derivative-energy ($E_{\text{energy}}(\varphi')$) and the value-range ($\varphi_H - \varphi_L$) of $\varphi(t)$, as required for the bit-allocation optimization.

**Solution:**

Here $\varphi_H = A$ and $\varphi_L = -A$, and the derivative energy is calculated as

$$E_{\text{energy}}(\varphi') = \int_0^1 (\varphi'(t))^2 \, dt = \int_0^1 (2\pi \omega A \cdot \cos(2\pi \omega t + \phi))^2 \, dt = 4\pi^2 \omega^2 A^2 \int_0^1 \cos^2(2\pi \omega t + \phi) \, dt$$

$$= 4\pi^2 \omega^2 A^2 \int_0^1 \frac{1}{2} (1 + \cos(4\pi \omega t + 2\phi)) \, dt = 2\pi^2 \omega^2 A^2 \left(1 + \frac{\sin(4\pi \omega t + 2\phi)}{4\pi \omega} \bigg|_0^1\right)$$

$$= 2\pi^2 \omega^2 A^2 \left(1 + \frac{\sin(4\pi \omega + 2\phi)}{4\pi \omega} - \frac{\sin(2\phi)}{4\pi \omega}\right) = 2\pi^2 \omega^2 A^2$$

where the last equality is due to the fact that $\omega$ is an integer and, therefore, $\frac{\sin(4\pi \omega + 2\phi)}{4\pi \omega} = \frac{\sin(2\phi)}{4\pi \omega}$.

b. Find the optimal number of samples ($N_t$) and quantization bits ($b$) under the constraint of overall bit-budget $B$.
   i. Formulate the bit-allocation optimization problem.
   ii. Develop the problem to optimization on a single variable (i.e., $b$ or $N_t$), and formulate the mathematical expression (equation) that defines optimality.

**Solution:**

i. The bit-allocation optimization is

$$\min_{N_t, b} \frac{1}{12N_t^2} E_{\text{energy}}(\varphi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}}$$

subject to $N_t b = B$
ii. We use the constraint to eliminate a variable:

\[ N_t = \frac{B}{b} \]

Then the optimization becomes

\[
\min_b \frac{b^2}{12B^2} \text{Energy}(\varphi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}}
\]

subject to \( b = \frac{B}{N_t} \)

the last optimization can also be written as (updating the constraints)

\[
\min_b \frac{b^2}{12B^2} \text{Energy}(\varphi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}}
\]

subject to \( 0 < b \leq B \)

\[ N_t = \frac{B}{b} \]

Let us optimize for \( b \) while ignoring the constraints (that will be later checked to hold):

\[
\min_b \frac{b^2}{12B^2} \text{Energy}(\varphi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}}
\]

Expression for optimality of \( b \) is obtained by demanding the derivative equality to zero:

\[
\frac{b}{6B^2} \text{Energy}(\varphi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}} \cdot (-2^{1-2b} \ln(2)) = 0
\]

that gives the optimality expression for \( b \):

\[
b \cdot 2^{2b} = \frac{(\varphi_H - \varphi_L)^2 \ln(2)}{\text{Energy}(\varphi')} \cdot B^2
\]

However, the last expression is analytically unsolvable.

c. You are given a total bit-budget of \( B = 200 \text{ bits} \). Compare (and explain) the values of the optimal bit-allocation parameters (i.e., \( N_t \) and \( b \)) of the following signals:

i. \( \varphi_1(t) = 5 \cdot \sin \left( 2\pi t + \frac{2}{3}\pi \right) \).
ii. \( \varphi_2(t) = 5 \cdot \sin \left( 20\pi t + \frac{2}{3}\pi \right) \).

Here you can use Matlab for numerically solving the optimality equation from section (b.ii). Useful Matlab functions here are `solve` and `lambertw`. Note that `solve` may return an expression that mathematically depends on the `lambertw` function, so, in turn, you can get a real valued result by appropriately applying the `lambertw` Matlab function.

**Solution:**

Numerical solution of the bit-allocation optimizations for \( \varphi_1(t) \) and \( \varphi_2(t) \) gives the corresponding solutions:

\[
\begin{align*}
b_1^{\text{opt}} &= 5.06 \\
b_2^{\text{opt}} &= 2.30
\end{align*}
\]

We observe that the solutions hold the constraint \( 0 < b \leq B \) that was temporarily ignored.

For practical bit-allocation we need integer values of \( b \). There is the closest integer from below, and the one from above. Note that the closer integer between the two does not necessarily provide the lower error. Accordingly, we need to evaluate the MSE for the two options:

Denoting the cost function as: \( MSE_B(b) = \frac{b^2}{12B^2} \text{Energy}(\varphi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^b} \)

For \( \varphi_1(t) \): \( MSE_B([b_1]) < MSE_B([b_1]) \) and therefore \( b_1^{\text{opt,int}} = 5 \).

For \( \varphi_2(t) \): \( MSE_B([b_2]) < MSE_B([b_2]) \) and therefore \( b_2^{\text{opt,int}} = 2 \).

The result that \( b_1^{\text{opt}} > b_2^{\text{opt}} \) aligns with the fact that \( \varphi_1(t) \) has lower frequency than \( \varphi_2(t) \), hence for \( \varphi_1(t) \) less samples are required allowing more bits for quantization.