Tutorial 13 – Part B

Wiener Filtering
Wiener Filtering for Signal Restoration: Discrete Problem Settings

A discrete signal, denoted as the $N$-length column vector $\phi$, is deteriorated according to the model:

$$\phi_{data} = H\phi + n$$

where

- $H$ is a known $N \times N$ matrix representing a linear degradation operator.
- $\bar{n}$ is an additive noise vector, considered as a realization of an $N$-length random vector, having i.i.d components that follow the properties:

  $$\mu_n \triangleq E\bar{n} = 0 \quad \text{and} \quad R_n \triangleq E\bar{n}\bar{n}^T = \sigma_n^2 I$$

- $\phi_{data}$ is the given degraded signal (a $N$-length column vector).

The task is to estimate the unknown signal $\phi$. 
Wiener Filtering for Signal Restoration: Discrete Problem Settings

The signal $\bar{\phi}$ is considered here as a realization of a random vector, associated with a class of signals, having the second-order statistics:

- Mean vector $\mu_{\phi} \triangleq E\{\bar{\phi}\} = 0$ (we consider zero mean for simplicity)
- An autocorrelation matrix $R_{\phi} \triangleq E\{\bar{\phi}\bar{\phi}^T\}$ (we consider real-valued signal for simplicity)

Then, the Wiener filter is the matrix

$$W = R_{\phi}H^T \left( HR_{\phi}H^T + \sigma_n^2 I \right)^{-1}$$

(we consider real-valued signal for simplicity)

and the signal estimate is

$$\bar{\phi}_{est}^{opt} = W\bar{\phi}_{data}$$

$$= R_{\phi}H^T \left( HR_{\phi}H^T + \sigma_n^2 I \right)^{-1}\bar{\phi}_{data}$$
Wiener Filtering for Signal Denoising: An Example

Consider a discrete signal, denoted as the $N$-length column vector $\bar{\varphi}$, being a realization of a (statistical) class of signals defined as:

$$\bar{\varphi} = [L_1, \ldots, L_1, L_2, \ldots, L_2]^T$$

where $L_1, L_2$ and $K$ are statistically independent random variables:

- $K$ is uniformly distributed among the integers $1, \ldots, N$.
- $L_1$ and $L_2$ follow $E\{L_1\} = E\{L_2\} = 0$ and $\text{var}\{L_1\} = \text{var}\{L_2\} = \sigma_L^2$. 

Wiener Filtering for Signal Denoising: An Example

The signal $\varphi$ is deteriorated by an i.i.d additive noise:

$$\varphi_{data} = \varphi + \bar{n}$$

where $\bar{n}$ follows the second-order statistics:

$$\bar{\mu}_n \triangleq E\bar{n} = \bar{0} \quad \text{and} \quad R_n \triangleq E\bar{n}\bar{n}^T = \sigma_n^2 I$$

and $\varphi_{data}$ is the given noisy signal.

• The task is to denoise $\varphi_{data}$ by estimating the unknown signal $\varphi$.

• Formulate the Wiener filter for the above defined denoising problem.
Wiener Filtering for Signal Denoising: An Example

Formulating the Wiener filter requires the second-order statistics of the signal class:

In general we write \( \vec{\varphi} = [\varphi_1, \ldots, \varphi_N]^T \)

then, the mean is

\[
E\{\vec{\varphi}\} = [E\{\varphi_1\}, \ldots, E\{\varphi_N\}]^T
\]

The mean of the \( i^{th} \) signal sample is

\[
E\{\varphi_i\} = E\{\varphi_i| i \leq K\} \cdot P\{i \leq K\} + E\{\varphi_i| i > K\} \cdot P\{i > K\}
\]

\[
= E\{L_1| i \leq K\} \cdot P\{i \leq K\} + E\{L_2| i > K\} \cdot P\{i > K\}
\]

(by the signal definition)

\[
= E\{L_1\} \cdot P\{i \leq K\} + E\{L_2\} \cdot P\{i > K\}
\]

(Since \( L_1 \) and \( L_2 \) are independent of \( K \))

\[
= 0 \cdot P\{i \leq K\} + 0 \cdot P\{i > K\}
\]

\[
= 0
\]

This implies that \( E\{\vec{\varphi}\} = \vec{0} \) (the signal has zero mean).
Wiener Filtering for Signal Denoising: An Example

The signal autocorrelation matrix is $\mathbf{R}_\phi = E\{\phi \phi^T\}$

and the $(i, j)$ component of $\mathbf{R}_\phi$ is $r_{ij} = E\{\phi_i \phi_j\}$.

Let us consider $i \leq j$:

$$E\{\phi_i \phi_j\} = E\{\phi_i \phi_j | i > K\} \cdot P\{i > K\}$$

$$+ E\{\phi_i \phi_j | j \leq K\} \cdot P\{j \leq K\}$$

$$+ E\{\phi_i \phi_j | i \leq K < j\} \cdot P\{i \leq K < j\}$$

$(i, j$ both to the right of the level-transition)

$(i, j$ both to the left of the level-transition)

(the level-transition is between $i$ and $j$)

We evaluate the three cases as:

$E\{\phi_i \phi_j | i > K\} = E\{L_2^2 | i > K\} = E\{L_2^2\} = \sigma_L^2$

$E\{\phi_i \phi_j | j \leq K\} = E\{L_1^2 | j \leq K\} = E\{L_1^2\} = \sigma_L^2$

$E\{\phi_i \phi_j | i \leq K < j\} = E\{L_1 L_2 | i \leq K < j\} = E\{L_1 L_2\} = E\{L_1\} E\{L_2\} = 0$

Leading to

$$E\{\phi_i \phi_j\} = \sigma_L^2 \cdot P\{i > K\} + \sigma_L^2 \cdot P\{j \leq K\}$$
We got that $E\{\varphi_i \varphi_j\} = \sigma_L^2 \cdot (P\{i > K\} + P\{j \leq K\})$

and due to the uniform distribution of $K$:

$$P\{i > K\} = \frac{i-1}{N} \quad \text{and} \quad P\{j \leq K\} = \frac{N-j+1}{N}$$

Then, for $i \leq j$:

$$E\{\varphi_i \varphi_j\} = \sigma_L^2 \cdot \left(\frac{i-1}{N} + \frac{N-j+1}{N}\right) = \sigma_L^2 \cdot \left(1 - \frac{j-i}{N}\right)$$

Similar developments for $i \geq j$ show:

$$E\{\varphi_i \varphi_j\} = \sigma_L^2 \cdot \left(1 - \frac{i-j}{N}\right)$$

Hence, for any $i, j$:

$$E\{\varphi_i \varphi_j\} = \sigma_L^2 \cdot \left(1 - \frac{|j-i|}{N}\right)$$
Wiener Filtering for Signal Denoising: An Example

Since  \( E\{\varphi_i \varphi_j\} = \sigma_L^2 \cdot \frac{N-|j-i|}{N} \) the signal autocorrelation matrix is

\[
R_{\varphi} = \sigma_L^2 \cdot \begin{bmatrix}
1 & \frac{N-1}{N} & \cdots & \frac{1}{N} \\
\frac{N-1}{N} & 1 & \ddots & \\
\vdots & \ddots & \ddots & \frac{N-1}{N} \\
\frac{1}{N} & \cdots & \frac{N-1}{N} & 1
\end{bmatrix}
\]

The Wiener filter, \( W \), for the described denoising problem (where \( H = I \)) is obtained by setting \( R_{\varphi} \) in

\[
W = R_{\varphi} \left( R_{\varphi} + \sigma_n^2 I \right)^{-1}
\]

The corresponding signal estimate (denoised signal): \( \varphi_{est}^{opt} = R_{\varphi} \left( R_{\varphi} + \sigma_n^2 I \right)^{-1} \varphi_{data} \)
Wiener Filtering for Signal Denoising: An Example

The corresponding signal estimate (denoised signal):

$$\varphi_{est}^{opt} = R_\varphi (R_\varphi + \sigma_n^2 I)^{-1} \varphi_{data}$$

where

$$R_\varphi = \sigma_L^2 \cdot \begin{bmatrix}
1 & \frac{N-1}{N} & \ldots & \frac{1}{N} \\
\frac{N-1}{N} & 1 & \ldots & \vdots \\
\vdots & \ddots & \ddots & \frac{N-1}{N} \\
\frac{1}{N} & \ldots & \frac{N-1}{N} & 1
\end{bmatrix}$$

What is the unitary matrix $U$ that transforms the denoising task to be componentwise?

Note that whereas $R_\varphi$ is Toeplitz, it is not circulant and therefore the DFT matrix does not diagonalize it.

Accordingly, we generally state that $U$ is the PCA matrix of $R_\varphi$. 

236327, CS Department, Technion, Spring 2018
Wiener Filtering for Signal Denoising: An Example

\[ \phi_{est}^{opt} = R_\phi \left( R_\phi + \sigma_n^2 I \right)^{-1} \phi_{data} \]

The transition into the PCA-domain component-based denoising is observed via:

\[ \phi_{est}^{opt,(U)} = U^{*} \phi_{est}^{opt} \]
\[ = U^{*} R_\phi \left( R_\phi + \sigma_n^2 I \right)^{-1} \phi_{data} \]
\[ = U^{*} R_\phi \left( UU^{*}(R_\phi + \sigma_n^2 I)UU^{*}\right)^{-1} \phi_{data} \]
\[ = U^{*} R_\phi \left( U^{*}(R_\phi + \sigma_n^2 I)U \right)^{-1} U^{*} U^{*} \phi_{data} \]
\[ = U^{*} R_\phi U \left( U^{*} R_\phi U + \sigma_n^2 U^{*} U \right)^{-1} U^{*} \phi_{data} \]
\[ = \Lambda_\phi \left( \Lambda_\phi + \sigma_n^2 I \right)^{-1} (U) \phi_{data} \]

(diagonal matrix --> component-based computation)

The solution \( \phi_{est}^{opt,(U)} \) should be transformed back to the signal domain via \( \phi_{est}^{opt} = U \phi_{est}^{opt,(U)} \)