Signal Processing, Representation, Modeling, and Analysis

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Chapter 8

Why Fourier?
or Operators on Representations
Operators can act on representations that are based on orthonormal families. Let the representation of \( \psi(t) : [0, 1) \rightarrow \mathbb{R} \) by the orthonormal basis \( \{ \beta_k(t) \} \) be given by

\[
\hat{\psi}(t) = \sum_{k \in \Omega} \psi_k^\beta \beta_k(t),
\]

for the set of indices \( k \in \Omega \).

Applying the linear operator \( \mathcal{H}_L \) to \( \hat{\psi}(t) \), yields

\[
\hat{\psi}^{\text{out}}(t) = \mathcal{H}_L(\hat{\psi}(t)) = \mathcal{H}_L \left( \sum_{k \in \Omega} \psi_k^\beta \beta_k(t) \right) = \sum_{k \in \Omega} \psi_k^\beta \mathcal{H}_L(\beta_k(t)),
\]

where we used the linearity of \( \mathcal{H}_L \).
\( \hat{\psi}^{\text{out}}(t) \) is a weighted combination of functions

\[
\{ \mathcal{H}_L(\beta_k(t)) \}_{k \in \Omega},
\]

which is not necessarily orthonormal. Indeed,

\[
\langle \mathcal{H}_L(\beta_k(t)), \mathcal{H}_L(\beta_l(t)) \rangle = \int_{\Delta} \int_{\rightarrow \infty}^{\leftarrow \infty} \beta_k(\xi) h_L(\xi, t) d\xi \left( \int_{\rightarrow \infty}^{\leftarrow \infty} \beta_l(\eta) h_L(\eta, t) d\eta \right)^* dt
\]

\[
= \int_{\Delta} \int_{\rightarrow \infty}^{\leftarrow \infty} \int_{\rightarrow \infty}^{\leftarrow \infty} \beta_k(\xi) \beta_l^*(\eta) h_L(\xi, t) h_L^*(\eta, t) d\xi d\eta dt.
\]

which is, in general, \( \neq \delta_{kl} \).

We would like to have

\[
\int_{\rightarrow \infty}^{\leftarrow \infty} \beta_k(\xi) h_L(\xi, t) d\xi = \lambda(k) \beta_k(t) \quad \forall k \in \Omega,
\]

that is,

\[
\mathcal{H}_L(\beta_k(t)) = \lambda(k) \beta_k(t),
\]

for all \( k \in \Omega \), with some numbers \( \lambda(k) \) in either \( \mathbb{R} \) or \( \mathbb{C} \).
If the functions $\beta_k(t)$ are invariant, up to a $k$ dependent weight, with respect to the operator $\mathcal{H}_L$, then each $\beta_k(t)$ is an eigenfunction of the operator $\mathcal{H}_L$, with eigenvalue $\lambda(k)$. In that case, the family $\{\mathcal{H}_L(\beta_k(t))\}_{k \in \Omega}$ satisfy

$$\langle \mathcal{H}_L(\beta_k(t)), \mathcal{H}_L(\beta_l(t)) \rangle = \int_{\Delta} \lambda(k) \lambda^*(l) \beta_k(t) \beta_l^*(t) dt$$

$$= \lambda(k) \lambda^*(l) \int_{\Delta} \beta_k(t) \beta_l^*(t) dt$$

$$= \lambda(k) \lambda^*(l) \delta_{kl} = \begin{cases} 0 & k \neq l \\ \lambda(k) \lambda^*(l) & k = l \end{cases}.$$ 

Next, we will search for the family of functions that satisfy

$$\mathcal{H}_L(\beta_k(t)) = \lambda(k) \beta_k(t).$$
Linear Shift Invariant operators are defined by

$$\mathcal{H}_{LSI}(\beta(t)) \equiv \int_{-\infty}^{\infty} \beta(\xi) h_{LSI}(t - \xi) d\xi,$$

we would like to also have

$$\int_{-\infty}^{\infty} \beta(\xi) h_{LSI}(t - \xi) d\xi = \lambda \beta(t),$$

or

$$\int_{-\infty}^{\infty} h(\xi) \beta(t - \xi) d\xi = \lambda \beta(t).$$

If $\beta(t - \xi) = \beta(t)f(\xi)$, then the above is satisfied. Functions that satisfy this condition are

$$e^{\alpha(t-\xi)} = e^{\alpha t} e^{-\alpha \xi}.$$
With this choice of functions

\[
\int_{-\infty}^{\infty} h(\xi) e^{\alpha(t-\xi)} d\xi = e^{\alpha t} \int_{-\infty}^{\infty} h(\xi) e^{-\alpha \xi} d\xi.
\]

For finite \( \int_{-\infty}^{\infty} h(\xi) e^{-\alpha \xi} d\xi = \lambda \), the function \( e^{\alpha t} \) is preserved by the convolution (up to the weight \( \lambda \)).

What would be the family of orthonormal functions \( \{\beta_k(t)\}_k \) satisfying that relation?

Obviously, \( \beta_k(t) = e^{f(k)t} \) would not do the work for any \( f(k) \), as

\[
\int_{\Delta} e^{f(k)t} (e^{f(l)t})^* dt = \int_{\Delta} e^{(f(k)+f(l))t} dt \neq 0.
\]

But ...
For the complex numbers $f(k) = i2\pi k$, we have

$$\int_{\Delta=[0,1)} e^{i2\pi kt} (e^{i2\pi kt})^* \, dt = \int_{\Delta=[0,1)} e^{i2\pi (k-l)t} \, dt = \delta_{kl} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}.$$ 

This is a “miracle.” The orthonormal family of functions \( \{e^{i2\pi kt}\}_{k \in \mathbb{Z}} \) that are mapped by linear operators into themselves belong to our good old Fourier family. An orthonormal basis of squared integrable functions over \([0,1)\).
The orthonormal Fourier family of complex functions \( \{e^{i2\pi kt}\}_{k \in \mathbb{Z}} \) are the eigenfunctions of any linear and shift invariant operator \( \mathcal{H}_L(\circ) \) of the form

\[
\psi^{\text{out}}(t) = \int_{-\infty}^{\infty} \psi^{\text{in}}(\xi) h_{LSI}(t - \xi) d\xi.
\]

Furthermore, we have

\[
\int_{-\infty}^{\infty} e^{i2\pi k\xi} h_{LSI}(t - \xi) d\xi = \int_{-\infty}^{\infty} h_{LSI}(\xi) e^{i2\pi k(t-\xi)} d\xi
\]

\[
= e^{i2\pi kt} \int_{-\infty}^{\infty} h_{LSI}(\xi) e^{-i2\pi k\xi} d\xi
\]

\[
= e^{i2\pi kt} \lambda_{\mathcal{H}_{LSI}}(k).
\]
The eigenvalues of $\mathcal{H}_L (\circ)$ corresponding to $\beta_k^F (t) = e^{i2\pi kt}$ are given by

$$\lambda_{\mathcal{H}_{LSI}} (k) = \int_{-\infty}^{\infty} h_{LSI}(\xi) e^{-i2\pi k\xi} d\xi.$$  

They form the Fourier Transform of the impulse response of $\mathcal{H}_{LSI}$ at the frequencies $k$ (in Hertz or oscillations on $[0, 1]$).

With this orthonormal family

$$\psi^{out} (t) = \mathcal{H}_{LSI} (\hat{\psi}(t)) = \mathcal{H}_{LSI} \left( \sum_{k \in \Omega} \psi_k^F e^{i2\pi kt} \right) = \sum_{k \in \Omega} \psi_k^F \lambda_{\mathcal{H}_{LSI}} (k) e^{i2\pi kt}.$$  

That is, applying $\mathcal{H}_{LSI}$ to $\hat{\psi}(t)$ requires multiplying its coefficients in the Fourier representation by the corresponding eigenvalues of $\mathcal{H}_{LSI}$. Amazing!
To $\infty$ and beyond

For

$$\int_{-\infty}^{\infty} h(\xi) e^{i2\pi \omega(t-\xi)} d\xi = e^{i2\pi \omega t} \int_{-\infty}^{\infty} h(\xi) e^{-i2\pi \omega \xi} d\xi,$$

we have the eigenfunctions property for $\{e^{i2\pi \omega t}\}_{\omega \in \mathbb{R}}$. We also have,

$$\int_{-\infty}^{\infty} e^{i2\pi \omega_1 t} e^{-i2\pi \omega_2 t} dt = \int_{-\infty}^{\infty} e^{i2\pi(\omega_1-\omega_2) t} dt$$

$$= \begin{cases} \int_{-\infty}^{\infty} 1 dt \to \infty & \omega_1 = \omega_2 \\ 0 & \omega_1 \neq \omega_2 \end{cases}$$

$$= \delta(\omega_1 - \omega_2).$$

That is, $\{e^{i2\pi \omega t}\}$ is a continuous orthonormal family over $\Delta = (-\infty, \infty)$. 
So far, we dealt with signals over $[0, 1)$ and periodically extended them beyond that. Thereby, the impulse response of $\mathcal{H}_L(\circ)$ given by $h(t) : [0, 1) \to \mathbb{R}$ can be periodically extended outside $[0, 1)$ such that $h(t - \xi)$ is well defined over $(-\infty, \infty)$.

![Fig. 8.11 - periodic $h(t)$.

Then,

$$
\int_0^1 h(\xi) e^{-\alpha(t-\xi)} d\xi = \left( \int_0^1 h(\xi) e^{\alpha \xi} d\xi \right) e^{-\alpha t},
$$

and for $\alpha = i2\pi$ we get our family of periodic orthonormal functions over $[0, 1)$ as desired.
With this Fourier family, for the input $\psi(t) : [0, 1) \to \mathbb{R}$ we get the output

$$\psi^{out}(t) = \int_0^1 \psi(\xi) h(t - \xi) d\xi,$$

and using

$$\psi(t) = \sum_{k=-\infty}^{\infty} \langle \psi(t), e^{i2\pi kt} \rangle e^{i2\pi kt} = \sum_{k=-\infty}^{\infty} \psi_k e^{i2\pi kt}$$

and

$$h(t) = \sum_{k=-\infty}^{\infty} \langle h(t), e^{i2\pi kt} \rangle e^{i2\pi kt} = \sum_{k=-\infty}^{\infty} h_k e^{i2\pi kt},$$

we have,
\[ \psi^{out}(t) = \int_{0}^{1} \psi(\xi) h(t - \xi) d\xi \]
\[ = \int_{0}^{1} \sum_{l} \sum_{k} \psi_{k} e^{i2\pi k \xi} h_{l} e^{i2\pi l(t-\xi)} d\xi \]
\[ = \sum_{l} \sum_{k} \psi_{k} h_{l} \int_{0}^{1} e^{i2\pi k \xi} e^{i2\pi l(t-\xi)} d\xi \]
\[ = \sum_{l} \sum_{k} \psi_{k} h_{l} e^{i2\pi l t} \int_{0}^{1} e^{i2\pi (k-l) \xi} d\xi \]
\[ = \sum_{k} \psi_{k} h_{k} e^{i2\pi kt}. \]

But,

\[ \psi^{out}(t) = \sum_{k} \psi^{out}_{k} e^{i2\pi kt} \quad \Rightarrow \quad \psi^{out}_{k} = \psi_{k} h_{k}. \]
Chapter Optimal Basis for Smooth Signals

Optimal Basis for Smooth Signals
We will provide the best representation basis (in terms of truncated representation) for a specific family of signals. Denote the family of functions $\psi : \Omega \rightarrow \mathbb{R}$ such that $\|\nabla \psi\|^2 \leq 1$ as $\{\psi_\omega\}$, and let $\{\beta_i\}$ be a basis over the smooth bounded domain $\Omega \subset \mathbb{R}^N$.

Again, let the squared representation error with the first $k$ elements of $\{\beta_i\}$ be defined as

$$E_k^2(\psi, \beta) = \left\| \psi - \sum_{i=1}^{k} \langle \psi, \beta_i \rangle \beta_i \right\|^2.$$

We would like to find the basis $\{\beta_i\}$ that minimizes $E_k^2(\psi, \beta)$ for all $k \geq 1$, for all $\psi \in \{\psi_\omega\}$.

We will prove that the eigenfunctions of the Laplace operator ordered by their corresponding eigenvalues in a non-decreasing order uniquely define the optimal basis.
Let $e = \{e_i\}$ be the orthonormal basis consisting the eigenfunctions of the Laplace operator

$$-\Delta e_i = \lambda_i e_i,$$

where we also assume that the eigenfunctions are equal to zero along the boundary of $\Omega$, i.e. $e_i = 0$ on $\partial \Omega$.

The ordered eigenvalues are $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. 
**Theorem 1** \( \forall k \geq 1 \) we have, \( \forall \psi \in \{ \psi_\omega \} \)

\[
\mathcal{E}_k^2(\psi, e) = \left\| \psi - \sum_{i=1}^{k} \langle \psi, e_i \rangle e_i \right\|_2^2 \leq \frac{\| \nabla \psi \|^2}{\lambda_{k+1}},
\]

**Proof.** One one hand,

\[
\left\| \psi - \sum_{i=1}^{k} \langle \psi, e_i \rangle e_i \right\|_2^2 = \left\| \sum_{i=k+1}^{+\infty} \langle \psi, e_i \rangle e_i \right\|_2^2 = \sum_{i=k+1}^{+\infty} \langle \psi, e_i \rangle^2
\]

on the other hand

\[
\| \nabla \psi \|^2 = \sum_{i=1}^{+\infty} \lambda_i \langle \psi, e_i \rangle^2 \geq \sum_{i=k+1}^{+\infty} \lambda_i \langle \psi, e_i \rangle^2 \geq \lambda_{k+1} \sum_{i=k+1}^{+\infty} \langle \psi, e_i \rangle^2
\]

We used the fact that

\[
\| \nabla \psi \|^2 \equiv \langle \nabla \psi, \nabla \psi \rangle = \langle \psi, -\Delta \psi \rangle + \infty \sum_{i=1}^{+\infty} \langle \psi, e_i \rangle e_i, \sum_{i=1}^{+\infty} \langle -\Delta \psi, e_i \rangle e_i \\
= \sum_{i=1}^{+\infty} \langle \psi, e_i \rangle \langle -\Delta \psi, e_i \rangle \\
= \sum_{i=1}^{+\infty} \langle \psi, e_i \rangle \langle \psi, -\Delta e_i \rangle \\
= \sum_{i=1}^{+\infty} \langle \psi, e_i \rangle \langle \psi, \lambda_i e_i \rangle \\
= \sum_{i=1}^{+\infty} \lambda_i \langle \psi, e_i \rangle^2
\]
Theorem 2 There is no $k \geq 1$ and no constant $0 \leq \alpha < 1$ and no ordered basis $\{\beta_i\}$, such that

$$
\left\| \psi - \sum_{i=1}^{k} \langle \psi, \beta_i \rangle \beta_i \right\|^2 \leq \frac{\alpha \|\nabla \psi\|^2}{\lambda_{k+1}} \quad \forall \psi \in \{\psi_\omega\}. 
$$

(1)

Proof. Poincaré’s trick [3]. Assume there is, and let

$$
\psi = c_1 e_1 + c_2 e_2 + \cdots + c_k e_k + c_{k+1} e_{k+1}. 
$$

The under-determined linear system

$$
\langle \psi, \beta_i \rangle = 0 \quad \forall i = 1, \cdots, k,
$$

of $k$ equations with $k + 1$ unknowns admits a non-trivial solution for $\{c_i\}$. Thus,

$$
\left\| \psi - \sum_{i=1}^{k} \langle \psi, \beta_i \rangle \beta_i \right\|^2 = \|\psi\|^2 = \sum_{i=1}^{k+1} c_i^2. 
$$

Inserting $\psi$ into (1) yields,

$$
\lambda_{k+1} \sum_{i=1}^{k+1} c_i^2 \leq \alpha \sum_{i=1}^{k+1} \lambda_i c_i^2 \leq \alpha \lambda_{k+1} \sum_{i=1}^{k+1} c_i^2.
$$

where we used the fact that $\|\nabla \psi\|^2 = \sum_{i=1}^{k+1} \lambda_i c_i^2$. Therefore, $\sum_{i=1}^{k+1} c_i^2 = 0 \Rightarrow c_i = 0$ for all $i$. A contradiction. \qed
**Uniqueness**

**Theorem 3** Let the orthonormal basis \( \{ \beta_i \} \) satisfy for all \( k \geq 1 \),

\[
\left\| \psi - \sum_{i=1}^{k} \langle \psi, \beta_i \rangle \beta_i \right\|^2 \leq \frac{\|\nabla \psi\|^2}{\lambda_{k+1}} \quad \forall \psi \in \{ \psi_\omega \}. \tag{2}
\]

Then \( \beta_i \equiv e_i \) with corresponding eigenvalues \( \lambda_i \).

**Proof.**

**Lemma 3.1** The basis signals for which (2) holds, satisfy

\( \langle \beta_j, e_l \rangle = 0 \) for \( 1 \leq j < l \).

**Proof.** (HW)

Complete the uniqueness proof for the case of a simple spectrum \( 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \).

Use Poincaré’s “magic trick”, follow the steps in [3] while simplifying for the case of a simple spectrum.
A Note about Dirichlet, Fourier and Laplace

Let \( \psi(t) : [0, 1) \to \mathbb{R} \) be a signal such that w.l.o.g. \( \psi'(0) = 0 \) and \( \psi'(1) = 0 \), also known as Neumann boundary conditions. The Dirichlet energy of \( \psi(t) \) is given by

\[
\| \nabla \psi(t) \|^2 \equiv \langle \nabla \psi(t), \nabla \psi(t) \rangle \\
= \int_0^1 \frac{d}{dt} \psi(t) \frac{d}{dt} \psi(t) dt \\
= \psi(t) \frac{d}{dt} \psi(t) \bigg|_{t=0}^1 - \int_0^1 \psi(t) \frac{d^2}{dt^2} \psi(t) dt \\
= \langle \psi(t), -\frac{d^2}{dt^2} \psi(t) \rangle \equiv \langle \psi(t), -\Delta \psi(t) \rangle.
\]

We could further solve for

\[
-\frac{d^2}{dt^2} e_i(t) = \lambda_i e_i(t).
\]

Did we say Fourier?
References


Chapter Functional Maps
Let $f(x) : [0, 1] \to \mathbb{R}$ be a smooth function. Let $\{\phi_i(x)\}_{i=1}^{\infty}$ be an orthonormal basis defined on $I = [0, 1]$, such that

$$\langle \phi_i(x), \phi_j(x) \rangle_I \equiv \int_{0}^{1} \phi_i(x)\phi_j(x)dx = \delta_{ij}, \quad (3)$$

where $\delta_{ii} = 1$ for all $i$, and $\delta_{ij} = 0$ for $i \neq j$.

We can represent $f(x)$ in the basis $\{\phi_i(x)\}$, so that,

$$f(x) = \sum_{i=1}^{\infty} \langle f(x), \phi_i(x) \rangle_I \phi_i(x)$$

$$= \sum_{i=1}^{k} \alpha_i \phi_i(x)$$

$$\approx \sum_{i=1}^{k} \alpha_i \phi_i(x). \quad (4)$$
Define a continuous one-to-one mapping between the interval $I = [0, 1]$ and the interval $\tilde{I} = [a, b]$ to be $T(x) : I \rightarrow \tilde{I}$.

We have $\tilde{x} = T(x)$ and $x = T^{-1}(\tilde{x})$. The function $\tilde{f}(\tilde{x}) : \tilde{I} \rightarrow \mathbb{R}$ can be constructed by using $T$ to translate the coordinates of $f(x)$, namely, $\tilde{f}(\tilde{x}) = f(T^{-1}(\tilde{x}))$.

Now, let $\{\tilde{\psi}_i\}_{i=1}^\infty$ be an orthonormal basis defined on $\tilde{I}$. We can express $\tilde{f}(\tilde{x}) = \sum_{i=1}^\infty \tilde{\beta}_i \tilde{\psi}_i(\tilde{x})$. It also holds that for each basis function $\phi_i(x)$ its mapping to $\tilde{I}$ is given by $\tilde{\phi}_i(\tilde{x}) = \phi_i(T^{-1}(x))$.

We readily have that

$$\tilde{\phi}_i(\tilde{x}) = \sum_{j=0}^\infty \langle \tilde{\phi}_i, \tilde{\psi}_j \rangle \tilde{\psi}_j(\tilde{x})$$

$$= \sum_{j=0}^\infty \tilde{\gamma}_{ij} \tilde{\psi}_j(\tilde{x}),$$

(5)

where we define $\tilde{\gamma}_{ij} \equiv \langle \phi_i(T^{-1}(\tilde{x})), \tilde{\psi}_j \rangle_{\tilde{I}}$. 
Finally, note that we could write $\tilde{f}(\tilde{x})$ as a function of $\tilde{\psi}(\tilde{x})$ and $\phi(x)$. Specifically,

$$
\tilde{f}(\tilde{x}) = \sum_{i=0}^{\infty} \langle f(x), \phi_i(x) \rangle \phi_i(T^{-1}(\tilde{x})) \\
= \sum_{i=0}^{\infty} \alpha_i \phi_i(\tilde{x}) \\
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \tilde{\gamma}_{ij} \tilde{\psi}_j(\tilde{x}) \\
= \sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} \alpha_i \tilde{\gamma}_{ij} \right) \tilde{\beta}_j \tilde{\psi}_j(\tilde{x}) \\
= \sum_{j=0}^{\infty} \tilde{\beta}_j \tilde{\psi}_j(\tilde{x}).
$$

(6)
We see that in order to *translate* the $\alpha$ coefficients into $\tilde{\beta}$’s, we have to integrate them using the inner product between the two bases, namely, $\tilde{\gamma}_{ij} = \langle \tilde{\phi}_i(\tilde{x}), \tilde{\psi}_j(\tilde{x}) \rangle_{\tilde{I}}$. But, note that the translated coefficients $\tilde{\gamma}_{ij}$ do not depend on the function we would like to express. Thus, the knowledge of $T$ makes the relation between the decomposition of corresponding functions on two different domains a linear operation. Again, the knowledge of $T$ allows us to define the inner product between the eigenfunctions on one domain and eigenfunctions defined on another domain. This inner product is the way to translate decomposition coefficients from one domain to another.
Interpretations

The above simple observation required the mapping $T$ in order to find the relation between two domains $I$ and $\tilde{I}$. The question is what can be done if $T$ is unknown, but, say $\tilde{\gamma}_{ij} \approx \delta_{ij}$. This is the case, up to sign ambiguities, in isometric asymmetric surfaces when using the eigenfunctions of the corresponding Laplace-Beltrami operators. So, given corresponding eigenfunctions, all that is left is find corresponding points. This can be accomplished, for example, by translating the decomposition of a Gaussian from $I$ to $\tilde{I}$ and then setting the corresponding maxima as corresponding points.
The situation is less favorable when $\tilde{\gamma}_{ij} \neq \delta_{ij}$ and there is a need to find the correspondence and the coefficients simultaneously. In that case, constraints of probable matches, and penalizing for $|\tilde{\gamma}_{ij} - \delta_{ij}|$, could lead to the desired matching function $T$.

Reference

Further thoughts

A trivial novel research would be to workout the functional maps with invariant eigenfunctions - specifically scale invariant. If you repeat the functional maps paper with scale invariant LBO you should be able to compare families of shapes as the scale-invariant LBO is ”less tight” compared to the regular one. In fact you could get correspondence with scale inv. LBO and then ”relax” and slowly ”shift” towards regular LBO by, say, tuning the modulation factor $\eta \in [0, 1]$, for $\tilde{g}_{ij} = |K|^\eta g_{ij}$. 
It would be interesting and illuminating to try that on geometric structures of shapes along the Darwin tree of evolution. Proving that for $\eta = 0$ one obtains identities, for $\eta = 0.4$ identity of species (union of all organisms capable of producing fertile offspring), and, say, $\eta = 1.1$ for family of species (like mammals). Note, that as $\eta$ gets larger, we “factor” the features into some sort of end points in the GH-space and limbs would be nothing but connections between these points.