LECTURE 8

WHY FOURIER? OR OPERATORS ON REPRESENTATIONS

(or how operators act on representations based on orthonormal families, and the eigenfunctions or invariant subspaces of operators).

8.1. Let us now consider the following:
Suppose we have a representation of a signal using an orthonormal family of functions

$$\hat{\psi}(t) = \sum_{k \in \mathbb{Z}} c_k \beta_k(t)$$

($k \in \mathbb{Z}$, a set of indices). Applying a transformation/operator \( \mathcal{H} \) to \( \hat{\psi} \) we get:
\[ \mathcal{X}_L \{ \phi(t) \} = \mathcal{X}_L \{ \sum_{k \in \mathcal{S}_2} \phi_k \beta_k(t) \} \]

and using linearity we have
\[ \mathcal{X}_L \{ \phi(t) \} = \sum_{k \in \mathcal{S}_2} \phi_k \cdot \mathcal{X}_L \{ \beta_k(t) \} \]

So we get that:
\[ \hat{\phi}_{\text{out}}(t) = \mathcal{X}_L \{ \phi(t) \} = \sum_{k \in \mathcal{S}_2} \phi_k \cdot \mathcal{X}_L \{ \beta_k(t) \} \]

i.e. \( \hat{\phi}_{\text{out}}(t) \) is a weighted superposition of functions from the family \( \{ \mathcal{X}_L \{ \beta_k(t) \} \}_{k \in \mathcal{S}_2} \).

However, this new family of functions is, in general, not orthonormal and we do not know what are its properties, in general.
Indeed

\[ \langle \mathcal{X}_k \beta_k(t), \mathcal{X}_l \beta_l(t) \rangle = \]

\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \beta_k(s) h_k(s,t) ds \int_{-\infty}^{+\infty} \beta_l(\eta) h_l(\eta,t) d\eta \] \ du dt

\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \beta_k(s) \beta_l^*(\eta) h_k(s,t) h_l^*(\eta,t) ds d\eta \] \ du dt

(in general) \neq \delta_{\ell k}!

For good things to happen we would like to have

\[ \int_{-\infty}^{+\infty} \beta_k(s) h_k(s,t) ds = \lambda(k) \beta_k(t) \]

for all \( k \in \mathbb{Z} \)

i.e. we would like to have:

\[ \mathcal{X} \{ \beta_k(t)^2 \} = \lambda(k) \beta_k(t) \]

for all \( k \in \mathbb{Z} \), with some numbers \( \lambda(k) \) (real or complex!)
Such desirable properties say that the functions $\beta_k(t)$ should be "almost" invariant under the operator $\mathcal{X}$, i.e., it should reproduce itself with perhaps a certain k-dependent weight $\lambda(k)$. In "sophisticated linear algebra" language, we want $\beta_k(t)$ to be eigenfunctions of the operator $\mathcal{X}$, with eigenvalues $\lambda(k)$.

If that is the case we'll have for the family $\{ \mathcal{X}_l \beta_k(t) \}$ the result

$$< \mathcal{X}_l \beta_k(t), \mathcal{X}_l \beta_k(t) > =$$

$$= \int \lambda(k) \hat{x}(l) \beta_k(t) \beta_k(t) dt =$$

$$= \lambda(k) \hat{x}(l) \int \beta_k(t) \beta_k(t) dt = \lambda(k) \hat{x}(l) \delta_{kl}$$

where

$$\delta_{kl} = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases}$$
Let us try to seek a family of functions obeying: \[ \mathcal{L}_2 \{ \beta(t) \} = \lambda(t) \beta(t) \]

8.2. **Linear and Shift Invariant Operators**

Here we have:

\[ \mathcal{L}_2 \{ \beta(t) \} = \int_{-\infty}^{+\infty} \beta(s) h_{LT}(t-s) \, ds \]

and we need to have:

\[ \int_{-\infty}^{+\infty} \beta(s) h_{LT}(t-s) \, ds = \lambda(t) \beta(t) \]

or:

\[ \int_{-\infty}^{+\infty} h(s) \beta(t-s) \, ds = \lambda \beta(t) \]

If we would have \( \beta(t-s) = \beta(t) \cdot f(s) \)

we'd be "in business". But we know a function that obeys this: indeed we have

\[ \beta(t-s) = e^{t-s} \]

\[ e^{(t-s)} = e^{t} e^{-s} \]
and with this choice:
\[ \int_0^\infty h(s) e^{(t-s)} \, ds = e^t \int_0^\infty h(s) e^{-s} \, ds \]

If the integral is finite then we got our wish; the function \( e^t \) reproduced itself in convolution and got weighted by \( \int_0^\infty h(s) e^{-s} \, ds \), a number we can call \( \lambda \) (if it is finite!)

This is all very nice but we would like to have a family of \( \lambda \) functions \( \{ \beta_k(t) \} \), all obeying such equations.

Clearly trying \( \beta_k(t) = e^{ft} \) does not work for any real \( f(t) \) since
\[ \int_\Delta e^{f(t)} (e^{ft})^t \, dt = \int e^{f(t) + fe(t)} \, dt \neq 0 \]
So we are led to try complex numbers, and we readily see that \( f(k) = i2\pi k \) yields

\[
\int_{\Delta=[0,1]} e^{i2\pi kt} (e^{i2\pi t})^x dt = \int_{\Delta=[0,1]} e^{i2\pi (k-1)t} dt =
\]

\[
= \delta_{k,l} = \begin{cases} 
1, & k=l \\
0, & k\neq l
\end{cases}
\]

"a miracle"

Indeed, a miracle occurred, not only did we find an orthonormal family of functions \( \{ e^{i2\pi k t} \} \), but we realize that we know this family very well: WE REDISCOVERED the Fourier family / an orthonormal basis for square-integrable functions over \([0,1]\). Let us summarize this beautiful result.
RESULT:

The orthonormal Fourier family of complex valued functions

\[ \{ e^{-i2\pi k t} \} \quad \text{for} \quad k \in \mathbb{Z} \]

are the eigenfunctions of any linear and shift invariant operator \( \mathcal{L}_{1,0,3} \) which has the form

\[
\phi(t) = \int_{-\infty}^{+\infty} e^{i2\pi k} h_{\mathcal{L}_{1,0,3}}(t-s) \, ds.
\]

Furthermore we have

\[
\int_{-\infty}^{+\infty} e^{i2\pi k s} h_{\mathcal{L}_{1,0,3}}(t-s) \, ds = \int_{-\infty}^{+\infty} h_{\mathcal{L}_{1,0,3}}(s) e^{i2\pi k(t-s)} \, ds = e^{i2\pi k t} \int_{-\infty}^{+\infty} h_{\mathcal{L}_{1,0,3}}(s) e^{-i2\pi k s} \, ds = e^{i2\pi k t} \mathcal{L}_{1,0,3}(k).
\]
The eigenvalues of \( \mathcal{H}_{1SI} \) corresponding to \( \beta_k^+(t) = e^{i2\pi kt} \) are given by

\[
\lambda_{\mathcal{H}_{1SI}}(k) = \int e^{-i2\pi k\xi} h_{1SI}(\xi) d\xi
\]

and they are the Fourier Transform of the impulse response of \( \mathcal{H}_{1TI} \) at the frequencies of \( k \) (in Hertz or oscillations per \([0, 1]\) interval!)

Using this on family we see that:

\[
\mathcal{H}_{1SI} \{ \mathcal{H}_{1SI} \{ \xi(t) \} \} = \mathcal{H}_{1SI} \{ \sum_k \xi_k e^{i2\pi kt} \}
\]

\[
= \sum_{k \in \mathbb{Z}} \xi_k \cdot \lambda_{\mathcal{H}_{1SI}}(k) \cdot e^{i2\pi kt}
\]

or in words: applying \( \mathcal{H}_{1SI} \) to \( \xi(t) \) just requires multiplying its coefficients in the Fourier representation by the corresponding eigenvalues of \( \mathcal{H}_{1SI} \): MARVELOUS!
Add here a discussion about the limits in the integrals $\Rightarrow i.e.$

\[
\int_{-\infty}^{\infty} f(s) e^{i2\pi ws} ds = e^{i2\pi wt} \int_{-\infty}^{\infty} f(s) e^{i2\pi ws} ds
\]

We have the eigenfunction properties for any $\{ e^{i2\pi wt} \}_{\text{Reals}}$. Indeed we also have that (formally):

\[
\int_{-\infty}^{\infty} e^{i2\pi wt} e^{-i2\pi w_2 t} dt = \begin{cases} 
1 & \text{if } w_1 = w_2 \\
\int_{-\infty}^{\infty} e^{i2\pi (w_1 - w_2) t} dt & \text{if } w_1 + w_2 = \delta(w_1 - w_2)
\end{cases}
\]

i.e. $\{ e^{i2\pi wt} \}$ form a continuous on family for $\Delta = (-\infty, +\infty)$. 

\[
\text{i.e. } \{ e^{i2\pi wt} \} \text{ form a continuous on family for } \\
\Delta = (-\infty, +\infty)
\]
However, in these lecture notes we deal with signals/ functions defined on $\Delta = [0, 1)$ and periodically extended outside the interval $\Delta$.

So an impulse response of $\mathcal{L}_\Delta \{ \cdot \}$ is $h(t) : [0, 1)$ and periodically extended outside $\mathbb{R}$.

Then we must have also that $h(t - \xi)$ is well defined everywhere in $(\mathbb{R})$.

Then
\[
\int_{0}^{1} h(\xi) e^{-\alpha (t-\xi)} \, d\xi = \left[ h(\xi) e^{\xi} \right]_{0}^{+\infty} e^{-\alpha t}.
\]

and for $\alpha = i 2\pi k$ we get a family of on periodic functions over $[0, 1)$, as derived.
With this family (which is Fourier) we have for an input \( \varphi(t) : [0,1] \to \mathbb{R} \) an output
\[
\varphi_{\text{out}}(t) = \int_0^1 \varphi(s) h(t-s) \, ds
\]
over \([0,1]\)

\[
= \int_0^1 \varphi(s) \cdot h(t-s) \, ds
\]

and using \( \varphi(t) = \sum_{k=-\infty}^{\infty} \langle \varphi(t), e^{i2\pi k t} \rangle e^{i2\pi k t} \) and \( h(t) = \sum_{k=-\infty}^{\infty} \langle h(t), e^{i2\pi k t} \rangle e^{i2\pi k t} \)
we have
\[
\int_0^1 \sum_{k=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \langle \varphi(t), e^{i2\pi k t} \rangle e^{i2\pi k t} \cdot \langle h(t), e^{i2\pi k t} \rangle e^{i2\pi k t} \, ds
\]

\[
= \sum_{k=-\infty}^{\infty} \varphi_k h_k \int_0^1 \langle e^{i2\pi k t}, e^{i2\pi k(s-t)} \rangle \, ds
\]

\[
= \sum_{k=-\infty}^{\infty} \varphi_k h_k \int_0^1 e^{i2\pi k t} \, ds
\]

\[
= \sum_{k=-\infty}^{\infty} \varphi_k h_k \int_0^1 e^{i2\pi k t} \, ds
\]
i.e.
\[ \psi_{\text{alt}}(+) = \sum \phi_k h e^{i \pi k t} \]

But:
\[ \psi_{\text{alt}}(+) = \sum \psi_k e^{i \pi k t} \]

\[ \Rightarrow \]
\[ \psi_k = \phi_k \cdot h \]

an important result!