LECTURE 7

FROM REPRESENTATIONS TO OPERATIONS

(Or the various operations can be carried out to modify/improve signals and the properties such operations may have; on linearity and shift invariance)

71 So far we have seen that given a signal \( f(t) : [0, 1) \rightarrow \mathbb{R} \cup \{ \pm \infty \} \), we can approximately represent it using families of orthonormal functions

\[ \{ \beta_1(t), \beta_2(t), \ldots, \beta_N(t) \} \]

and, in fact, map the "continuous" \( f(t) \) into finite sets of coefficients \( \{ c_1, c_2, \ldots, c_N \} \) which yield approximations of the form
\[ \hat{\phi}_N^B(t) = \sum_{k=1}^{N} \phi_k^B \beta_k(t) \]

In these representations we could measure the error \( \varepsilon_N^B(t) = \phi(t) - \hat{\phi}_N^B(t) \) by the mean squared error

\[
\text{MSE} = \int_0^1 \| \varepsilon_N^B(t) \|^2 dt = \sum_{k=1}^{N} \| \phi_k^B \|^2
\]

and these formulae were correct for all the families we have seen so far.

1) \[ \{ \Delta_k(t) \}_{k=1,2,\ldots,N} \text{ yielding } \phi_1^s \phi_2^s \ldots \phi_N^s \]
2) \[ \{ \text{haar}_k(t) \}_{k=1,2,\ldots,N-2} \text{ yielding } \phi_1^h \phi_2^h \ldots \phi_{N-2}^h \]
3) \[ \{ \text{haar}_k(t) \}_{k=1,2,\ldots,N-2} \text{ yielding } \phi_1^har \phi_2^har \ldots \phi_{N-2}^har \]
4) \[ \{ \text{Fourier}_k(t) \}_{k=1,2,\ldots,M=2N+1} \text{ yielding } \phi_1 \phi_2 \ldots \phi_M \]
(In all these cases the approximations obtained with all the elements used in the superposition for $P^k(t)$ being the same as the basic standard piecewise constant approximation)

and

$$5) \{ e^{i2\pi t k} \}_{k=-N_0,N_0-1, \ldots ,0,1,2, \ldots ,N_0}$$

yielding $\{ e_0^F, e_1^F, \ldots , e_{-N_0}^F, e_{-N_0}^F, \ldots , e_N^F \}$ which yields a continuous (not piecewise constant) approximation, different from the 4 cases above.

Here we considered the case of band-limited $p(t)$, with nonzero $e_k^F$'s only for $|k|<N_0$.

and showed that having $2N_0+1$ uniformly spaced samples of $p(t)$ enables us to recover $p(t)$ exactly via the DFT matrix (providing the coefficients $\{ e_k^F \}$ from the samples.)
This result, by the way is known in the world as the Nyquist sampling theorem (in a slightly different setting, for functions defined over \((-\infty, \infty)\) and having bandlimited continuous Fourier representations).

Then we considered a smoothed version of \(e(t)\) via an operator

\[
\text{Smooth}\{e(t)\} = \frac{1}{\Delta t} \int_{t}^{t+\Delta} e(\xi) \, d\xi = \phi^\text{smooth}(t)
\]

and showed that its samples are exactly the values of \(f_k^S\) for \(k = 1, 2, \ldots, M (= 2N+1)\)

if multiplied by \(\frac{1}{M}\), and these \(M\) equispaced and weighted samples of \(e^\text{smooth}(t)\) enable its full recovery over \([0, 1)\) via a DFT yielding the Fourier coefficients of \(e^\text{smooth}(t)\).
Hence DFT \[
\begin{bmatrix}
\mathbf{e}_1^s \\
\mathbf{e}_2^s \\
\vdots \\
\mathbf{e}_M^s
\end{bmatrix}
\] where \(\{\mathbf{e}_k^s\}\) are the standard representation coefficients of the original \(\mathbf{e}(t)\) yield the complete recovery of Smooth \(\mathbf{e}(t)\) to \(\mathbf{e}^\text{smooth} (t)\).

In other words, if we replace

\[
\mathbf{e}_M^\text{Fourier} (t) = \sum_{k=-N_0}^{N_0} \mathbf{e}_k \cdot \mathbf{e}_k^\text{Fourier} (t)
\]

with \(\sum_{k=1}^{M} (\mathbf{e}_k^s \sqrt{M} A(t))\), the Fourier \(\mathbf{e}(t)\) functions by \(e^{i2\pi ft}\) we get

\[
\mathbf{e}(t) = \sum_{k=-N_0}^{N_0} \mathbf{e}_k \cdot e^{i2\pi ft}
\]

\[
\equiv \sum \mathbf{e}_k^\text{Smooth} \cdot e^{i2\pi ft}
\]

\[
\equiv \mathbf{e}^\text{smooth} (t)
\]

if \(\mathbf{e}(t)\) is BandLimited.
Therefore the mean squared error in using the coefficients \( \hat{c}_k \) for \( k = 1, 2, \ldots, N \) and the functions \( \cos(2\pi kt) \) for \( k = 0, 1, \ldots, N_0 - N_0 \ldots - 1 \) in approximating \( f(t) \) will be

\[
\psi_{MSE}(f(t) \rightarrow \sum \hat{c}_k \cos(2\pi kt)) = \\
= \int_0^1 (f(t) - \hat{f}(t))^2 dt = \\
= \int_0^1 (f(t) - \frac{1}{N_0} \int_0^{T \Delta} f(s) ds)^2 dt = \\
= \int_0^1 (\sum_{k=-N_0}^{N_0} \hat{c}_k e^{2\pi i k t} - \sum_{-N}^{N} \hat{c}_k e^{2\pi i k t})^2 = \\
= \int_0^1 \left( \sum_{k=-N_0}^{N_0} (\hat{c}_k - \hat{c}_k) e^{2\pi i k t} \right)^2 = \\
= \sum_{-N_0}^{N_0} \left( \hat{c}_k - \hat{c}_k \right)^2 = \sum_{-N_0}^{N_0} \left\| \hat{c}_k \right\| \left( 1 - e^{2\pi i k N_0 / T \Delta} \right)^2
\]
\[
\psi_{MSE} = \sum_{-N_0}^{N_0} \| \varphi_k \|^2 \| \frac{(2\pi k \Delta A - \sin 2\pi k \Delta A)}{2\pi k \Delta A} - (1 - \cos 2\pi k \Delta A)^2 \| \frac{1}{2} = \\
(\| A \cdot B \|^2 = \| A \|^2 \| B \|^2) \\
= \sum_{-N_0}^{N_0} \| \varphi_k \|^2 \frac{1}{4\pi^2 k^2 \Delta A^2} \left( 4\pi^2 k^2 \Delta A^2 - 4\pi k \Delta A \sin 2\pi k \Delta A + \sin^2 \right) \\
+ 1 - 2 \cos 2\pi k \Delta A + \cos^2 \right) \\
= \sum_{-N_0}^{N_0} \| \varphi_k \|^2 \frac{1}{4\pi^2 k^2 \Delta A^2} \left( 4\pi^2 k^2 \Delta A^2 - 4\pi k \Delta A \sin(2\pi k \Delta A) \\
+ 2 - 2 \cos(2\pi k \Delta A) \right) \\
= \sum_{-N_0}^{N_0} \| \varphi_k \|^2 \left( 1 - 2 \frac{\sin(2\pi k \Delta A)}{2\pi k \Delta A} \right) + \\
+ 2 \frac{1 - \cos(2\pi k \Delta A)}{4\pi^2 k^2 \Delta A^2} \\
\text{(as } \Delta A \to 0 \text{ we have by L'Hopital's rule:)} \\
1 - 2 \frac{1}{2} + 2 \frac{1}{2} = 1 - 2 + 1 = 0
\]
We obtained that:

\[
\psi_{\text{MSE}}(\Phi(t)) = \sum_{-N_0}^{N_0} (\psi_{\text{smooth}}(t) e^{i2\pi k t})^2 (1 + \left(\frac{\sin \pi k/\Delta}{\pi k/\Delta} - 2 \cos k\theta \right) \frac{\sin \pi k/\Delta}{\pi k/\Delta})
\]

And note that as \( \Delta \to 0 \), while \( N_0 \) increases the terms multiplying \( ||\psi_k||^2 \) decrease to zero.

Actually as \( N_0 \to \infty \) and \( \Delta \to 0 \) we can evaluate also

\[
\psi_{\text{MSE}}(\Phi(t)) = \int_0^{\frac{\pi}{\Delta}} \psi(\xi) d\xi
\]

\[
= \int_0^1 (\Phi(t) - \frac{1}{1\Delta} \int_0^{t+1\Delta} \psi(\xi) d\xi) dt
\]

\[
= \int_0^1 (\Phi(t) - \frac{1}{1\Delta} \int_t^{t+1\Delta} (\Phi(t) + \Phi(t) \xi) d\xi) \int_0^1 dt
\]

\[
= \int_0^1 (\Phi(t) - \frac{1}{1\Delta} \int_0^{1\Delta} (\Phi(t) + \Phi(t) \xi) d\xi) dt
\]

\[
= \int_0^1 (\Phi(t) - \frac{1}{1\Delta} \int_0^{1\Delta} (\Phi(t) + \Phi(t) \xi) d\xi) dt
\]

\[
= \int_0^1 (\Phi(t) - \frac{1}{1\Delta} \int_0^{1\Delta} (\Phi(t) + \Phi(t) \xi) d\xi) dt
\]
\[ \sum_{\text{MSE}} (\hat{\theta}(t) - \theta^\text{smooth}(t)) = \]
\[ = \int_{0}^{1} (\hat{\theta}(t) \Delta(t + \Delta)) \frac{\Delta^2}{2} dt = \]
\[ = \Delta^2 \int_{0}^{1} (\hat{\theta}(t) \Delta^2) dt + O(\Delta^3) \]
In the analysis carried out above, we encountered an "operator" applied to $\phi(t)$ which produced another function $\phi'(t)$. Actually, in signal processing we are often required to design "operators" that do something useful for us, in the sense of producing for our senses or for analysis "better" signals. Here by better we mean:

- cleaner, i.e. less "noisy"
- enhanced, i.e. "sharper"
- restored, i.e. closer to some original desired signal that was recorded with "distortion", or underwent some "degradation".
Next, we shall consider various types of 
"operators" that may be applied to signals 
(or images, or videos, or data sequences, or 
"time series", or vectors).

As usual we start with a given 
signal $f(t) \in [0, 1) \rightarrow [e^{-}, e^{+}]$
that will be subjected to all sorts 
of "operator" indignities to produce 
"output" signals $g(t)$ as a result.

(We shall assume that beyond $[0, 1)$ the 
signal is extended periodically as 
the Fourier representations implicitly 
do). We shall describe the process 
of applying an operator $\mathcal{H}$ to a function 
$f(t)$ as follows:
\[ \phi_{\text{out}}(t) = \mathcal{H}\{\phi_{\text{in}}(t)\} \]

or

\[
\begin{array}{c}
\phi_{\text{in}}(t) \\
\text{THE INPUT}
\end{array} \xrightarrow[]{} \mathcal{H} \xrightarrow[]{} \phi_{\text{out}}(t)
\]

As we shall see in the examples below, the operator (or "system", or "operation" or "signal processing algorithm") $\mathcal{H}$ generates an output, or a result, which is also a signal, just like the input.

Examples of operators:

1. \[ \phi_{\text{out}}(t) = \frac{1}{|\Delta|} \int_{t-\frac{\Delta}{2}}^{t+\frac{\Delta}{2}} \phi(\tau) \, d\tau \]

   This is a "moving average", smoothing operator (we have seen a slightly different one before!)
2) \[ C_{\text{out}}(t) = C^2(t) \] or in general

\[ C_{\text{out}}(t) = \mathbb{L}[C(t)] \] where \( \mathbb{L}[\cdot] \)

is some general \( \mathbb{R} \to \mathbb{R} \) map.

These are \( \mathbb{R} \) (nonlinear) maps look up-table or "point-to-point" functions in which \( C_{\text{out}}(t) \) depends only on \( C_{\text{in}}(t_0) \). Another words these are local memoryless maps.

3) \[ C_{\text{out}}(t) = \int_{-\infty}^{\infty} W(\xi, t) C(\xi) d\xi \]

This type of operation produces a weighted average of all \( C_{\text{in}}(t) \)'s where the weight function depends on \( t \).

\( W(\xi, t) \) can be, and often is a shifted version of a given function \( h(\xi) \)

i.e. \( W(\xi, t) = h(\xi - t) \)
Note that $h(\xi-t)$ has the origin value ($h(\xi=0)$) moved to $\xi=t$!

Sometimes we shall also use

$$w(\xi,t) = h(t-\xi)$$

Note that here too the origin is moved to $\xi=t$, however the function is first flipped from $h(\xi)$ to $h(-\xi) = \bar{h}(\xi)$. Note that $\bar{h}(\xi-t) = h(-(\xi-t)) = h(t-\xi)$. The operation

$$\phi(\xi)(t) \overset{\text{out}}{=} \int_{-\infty}^{+\infty} \phi(\xi) h(t-\xi) \, d\xi$$

is very important, as we shall see, and is called a convolution.
4) \[ \phi^{\text{out}}(t) = \phi^{\text{in}}(t+T) \]

This operation is called a time shift or delay/advance operator.

Note that \( \phi^{\text{in}}(0) \) is moved to \( \phi^{\text{in}}(T) \) i.e. to a later time +T or an earlier one -T.

This is a very important operator and we shall write it as follows:

\[ \phi^{\text{out}}(t) = T \left\{ \phi^{\text{in}}(t) \right\} = \phi^{\text{in}}(t-T) \]

The time shift operator is parameterized by T.
5) \( \phi_{\text{out}}(t) = \phi_{\text{in}}(t) + m(t) \)

\( \phi_{\text{out}}(t) = m(t) \cdot \phi_{\text{in}}(t) \)

These operations add or multiply the input with a given function \( m(t) \) independent of \( \phi(t) \).

So far we have seen various types of operations that we may perform on a given input \( \phi(t) \) to produce the output function \( \phi_{\text{out}}(t) \). These operations have different characteristics and properties and our task is to analyse their effect and to design various operators that achieve desired effects. Most importantly...
7.3 Properties of operations.

In representing functions with the help of various orthonormal families of known and designed functions multiplied by constant coefficients we have seen the importance of considering functions as superpositions or sums of other functions. One therefore naturally raises the question how does an operator act on weighted sums of functions.

Linear Operators

An operator \( \mathbf{X} \) will be called linear if

\[
\mathbf{X}\{a \varphi_1(t) + b \varphi_2(t)\} = a \mathbf{X}\{\varphi_1(t)\} + b \mathbf{X}\{\varphi_2(t)\}
\]

for any two functions \( \varphi_1(t), \varphi_2(t) \) and any scalars \( a, b \).
From the basic definition we have that linear operators obey:

\[ \mathcal{H} \{ \sum_k x_k \phi_k(t) \} = \sum_k \mathcal{H} x_k \phi_k(t) \]

and also

\[ \mathcal{H} \{ \int_\Omega x(\xi) \gamma(\xi, t) d\xi \} = \int_\Omega x(\xi) \mathcal{H} \gamma(\xi, t) d\xi \]

and here \( x(\xi) \) is a function of coefficients and integration is performed over a given range \( \Omega \) of \( \xi \).

**Shift/Translation Invariant Operators**

An operator \( \mathcal{H} \) will be called shift invariant if we have

\[ \mathcal{H} \{ \mathcal{T}_{t_0} \{ \phi(t) \} \} = \mathcal{T}_{t_0} \{ \mathcal{H} \{ \phi(t) \} \} \]

i.e. if it commutes with the shift operator

\[ \mathcal{T}_{t_0} \{ \phi(t) \} = \phi(t-t_0) \] for all \( t_0 \).
Shift Invariant operators obey
\[ \mathcal{H}_{\text{SI}} \{ \phi(t) \} = e^{\text{at}} (t) \]
\[ \Rightarrow \mathcal{H}_{\text{SI}} \{ \phi(t-t_0) \} = e^{\text{at}} (t-t_0) \text{ for all } t_0. \]

Let us consider the Examples we had before and see whether the operators are linear and or shift invariant.

**Notation:** An operation/operator/system \( \mathcal{H} \)
that is linear will be denoted \( \mathcal{H}_L \).
An operator that is shift invariant will be denoted \( \mathcal{H}_{\text{SI}} \).
An operator that is both linear and shift invariant will be denoted \( \mathcal{H}_{L\text{SI}} \).
Example 1: We have
\[ \mathcal{F}(\phi) = \mathcal{H}\{\phi(t)\} = \frac{1}{\sqrt{A}} \int_{t-A/2}^{t+A/2} \phi(s) ds \]
\[ = \int_{-\infty}^{+\infty} \phi(s) \cdot \frac{1}{\sqrt{A}} \mathbf{1}_A(t-s) ds \]

where
\[ \mathbf{1}_A(t) = \begin{cases} 1 & t \in [-A/2, A/2] \\ 0 & t \notin [-A/2, A/2] \end{cases} \]

\[ \mathbf{1}_A(t-s) = \begin{cases} 1 & \text{if } t-s \in [-A/2, A/2] \\ 0 & \text{otherwise} \end{cases} \]

\[-A/2 \leq t-s \leq A/2 \implies \begin{cases} s \leq t+A/2 \\ s \geq t-A/2 \end{cases} \]

The operator is a weighted integration of known functions, where the weights are determined by the input function. We could rewrite \( \mathcal{F}(\phi) \) as follows:
\[ \mathcal{F}(\phi) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{A}} \mathbf{1}_A(s) \phi(t-s') ds' \]

\[ t-s = s' \implies s = t-s' \]
Clearly if we put \( a \phi_1(t) + b \phi_2(t) = \phi_{\text{input}}(t) \)
we shall obtain
\[
\phi_{\text{out}}(t) = a \phi_1(t) + b \phi_2(t)
\]
Hence the operator is linear.

**Shift Invariance:** If we replace \( \phi_{\text{in}}(t) \) by 
\( \phi_{\text{in}}(t-t_0) \) for a given \( t_0 \), we obtain
\[
\tilde{\phi}_{\text{out}}(t) = \int_{-\infty}^{+\infty} \phi(\xi-t_0) \frac{1}{|A|} \Delta (t-t_0 - \xi) d\xi = 
\]
and defining \( \xi - t_0 = \xi_{\text{new}} \) we get
\[
= \int_{-\infty}^{+\infty} \phi(\xi_{\text{new}}) \frac{1}{|A|} \Delta (t-t_0 - \xi_{\text{new}}) d\xi_{\text{new}}
\]
\( \Delta \phi_{\text{out}}(t-t_0) \)
Hence the smoothing operator is also shift-invariant.
Example 2

The local "look-up-table" maps

\[ \phi(t) = \mathcal{L}[\phi(t)] \]

We have \( \mathcal{L}[a \phi(t) + b \psi(t)] = a \mathcal{L}[\phi(t)] + b \mathcal{L}[\psi(t)] \)

if and only if \( \mathcal{L} \) is linear, but in general it is not. For \( \mathcal{L}[x] = x^2, \sin x, e^x \) etc it will not be linear. For \( \mathcal{L}[x] = \text{polynomial} \) it will be linear while for \( \mathcal{L}[x] = ax + b \) it will be affine but not linear.

Clearly we have

\[ \mathcal{L} \left[ \mathcal{T}_{t_0} \{ \phi(t) \} \right] = \mathcal{T}_{t_0} \left[ \mathcal{L}[\phi(t)] \right] = \mathcal{T}_{t_0} \{ \mathcal{L} \left[ \phi(t) \right] \} \]

hence we have that this operator is shift-invariant.
Example 3

Clearly the averaging process is linear since:

$$\int_{-\infty}^{\infty} W(\xi, t) \left[ a \phi(\xi) + b \varphi(\xi) \right] d\xi =$$

$$= a \int_{-\infty}^{\infty} W(\xi, t) \phi(\xi) d\xi + b \int_{-\infty}^{\infty} W(\xi, t) \varphi(\xi) d\xi$$

but it is not shift invariant, unless

$$W(\xi, t) = h(\xi - t).$$

Example 4

$$\mathcal{T}_T \{ a \phi(t) + b \varphi(t) \} = a \mathcal{T}_T \{ \phi(t) \} + b \mathcal{T}_T \{ \varphi(t) \}$$

hence the shift operator is linear and it is also shift invariant.

$$\mathcal{T}_T \{ \mathcal{T}_{t_0} \{ \phi(t) \} \} = \mathcal{T}_T \{ \mathcal{T}_{t_0} \{ \phi(t) \} \}$$

$$= \phi(t - t_0 - T) \equiv \phi(t - T - t_0) \text{ for all } T, t_0$$
Example 5

Since \( a\varphi_1(t) + b\varphi_2(t) + m(t) \neq a(\varphi_1(t) + m(t)) + b(\varphi_2(t) + m(t)) = a\varphi_1(t) + b\varphi_2(t) + (a+b)m(t) \)

for all \(a, b\), this operation is \textbf{NOT LINEAR}.

Since \( \varphi(t-t_0) + m(t-t_0) \neq \varphi(t-t_0) + m(t) \)

the operation is not \textbf{SHIFT INVARIANT}.

Also since

\[ m(t)(a\varphi_1(t) + b\varphi_2(t)) = a(m(t)\varphi_1(t)) + b(m(t)\varphi_2(t)) \]

this operation is \textbf{LINEAR}, but since

\[ m(t)\varphi(t-t_0) \neq m(t-t_0)\varphi(t-t_0) \]

this operation is not \textbf{SHIFT INVARIANT}.
7.4. Dirac’s $\delta$-function and Linear Operators

Let us reconsider the moving average smoothing operator. Given $\phi(t)$ we have

$$\phi_{\frac{1}{\Delta}}(t) = \frac{1}{|\Delta|} \int_{t-\frac{1}{\Delta}}^{t+\frac{1}{\Delta}} \phi(s) \, ds = \int_{-\infty}^{+\infty} \phi(s) \frac{1}{|\Delta|} \chi_{[-\frac{1}{\Delta}, \frac{1}{\Delta}]}(t-s) \, ds$$

As $|\Delta| \to 0$ we shall clearly have (for nice $\phi(t)$’s that are smooth (e.g., differentiable)) that

$$\phi_{\frac{1}{\Delta}}(t) \to \phi(t) \quad \text{as} \quad |\Delta| \to 0$$

This is easy to see: $\phi(s)$ over $[t-\frac{1}{\Delta}, t+\frac{1}{\Delta}] = \Delta$

is $\phi(s) \approx \phi(t) + \phi'(t)(s-t) + O(|\Delta|)$

and $\phi_{\frac{1}{\Delta}}(t) = \phi(t) + \phi'(t) \int_{t-\frac{1}{\Delta}}^{t+\frac{1}{\Delta}} (s-t) \, ds + O(|\Delta|)$

$$= \phi(t) + 0 + O(|\Delta|) \to \phi(t)$$

Consider

$$\frac{1}{|\Delta|} \int \phi(t) = \begin{cases} \frac{1}{|\Delta|} & t \in \Delta \\ 0 & t \notin \Delta \quad \text{as} \quad |\Delta| \to 0 \end{cases}$$

($\Delta = [-\frac{1}{2\Delta}, \frac{1}{2\Delta}]$ a self-referencing but clear definition.)
Let us define the limiting function as \( \delta(t) \). Interestingly, this "function" is zero everywhere except at \( t=0 \), where it becomes infinite (\( \frac{1}{|a|} \) as \( |a| \to 0 \)) but has the following properties:

\[
\int_{-\infty}^{+\infty} \delta(t) \, dt = 1
\]

and, most importantly, we have for \( \delta(t) \) the following "defining" property

\[
\int_{-\infty}^{+\infty} \varphi(\xi) \delta(t-\xi) \, d\xi = \varphi(t)
\]

and

\[
\int_{-\infty}^{+\infty} \varphi(\xi) \delta(t+\xi) \, d\xi = \varphi(t)
\]

Hence the "function" \( \delta(t) \), which is called the Dirac \( \delta \)-function or the Heaviside \( \delta \),
reproduces the function $f(t)$ as a superposition of weighted shifts of $\delta(t)$, where the weight for $\delta(t-\xi)$ is $p(\xi)$. Note that $\delta(t-\xi)$ is the same as $\delta(\xi-t)$.

Some "cute" properties of $\delta(t)$ are

1) $\delta(x+t) = \frac{1}{|x|} \delta(t) \quad (\Rightarrow \delta(-t) = \delta(t))$

2) $\delta(g(t)) = \sum_{x_i} \frac{1}{|g'(x_i)|} \delta(t-x_i)$
   where $x_i$ are the simple roots of $g(x)$

3) $\int_{-\infty}^{+\infty} g(t) \delta(t-t_0) \, dt = g(t_0)$
   the Sifting property

4) $\int_{-\infty}^{+\infty} f(t) \delta(g(t)) \, dt = \sum_{x_i} \frac{1}{|g'(x_i)|} f(x_i)$

5) $\int_{-\infty}^{+\infty} \delta(\xi) \, d\xi = \text{Step}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$
   so in a sense $\frac{d}{dt} \text{Step}(t) = \delta(t)$
The most important fact concerning the Dirac δ-function is that

\[ \varphi(t) = \int_{-\infty}^{\infty} \varphi(\xi) \delta(t-\xi) d\xi = \left( \int_{-\infty}^{\infty} \varphi(\xi) \delta(\xi-t) d\xi \right) \]

This is a "representation" formula that exhibits \( \varphi(t) \) as a (continuous) superposition of functions of \( t \) parametrized/indexed by \( \xi \), i.e. \( \{ \delta(t-\xi) \}_{\xi \in \mathbb{R}} \) and weighted by \( \varphi(\xi) \), very similar to representations like

\[ \varphi(t) = \sum_{k} \varphi_{k} \beta_{k}(t) \]

w.r.t. families of functions \( \{ \beta_{k}(t) \}_{k \in \mathbb{Z}} \)
In what sense, we may ask, is the family of displaced/shifted δ-functions \( \{ \delta(t-x) \}_{x \in \mathbb{R}} \) an on-like family? Well, formally putting \( \Phi(t) = \delta(t-n) \) in the representation formula we obtain that

\[
\int_{-\infty}^{\infty} \delta(x-n) \delta(x-t) \, dx = \delta(t-n)
\]

and this is just like:

\[
\int_{-\infty}^{\infty} \beta_k(x) \beta_l(x) \, dx = \delta_{k,l}
\]

for the case of families of on-functions indexed by integers.

Hence the analogy is quite remarkable!
For us the most important fact is the way in which $\delta(t)$ enables us to write

$$p(t) = \int_{-\infty}^{+\infty} y(\xi) \delta(t-\xi) d\xi$$

because we here have a representation of $p(t)$ as a superposition of standard functions $\{\delta(t-\xi)\}_{\xi \in \mathbb{R}}$ multiplied by coefficients which are just the values of $y(\xi)$ at $\xi$'s. In some sense the $\{\delta(t-\xi)\}_{\xi}$ form a good family for function representation in the same way as $\{\mathbb{1}_A(t)\}_{A \in \mathbb{R}} = \{\mathbb{1}_A(t-\xi)\}_{\xi \in \mathbb{R}}$.

Indeed we have that the $\{\delta(t-\xi)\}_{\xi}$ family behave like an orthonormal family of functions and from the representation (convolution) formula we can write putting $\psi(t) \triangleq \delta(t-\eta)$ that:

$$\int_{-\infty}^{+\infty} \psi(\xi - \eta) \delta(t-\xi) d\xi = \delta(t - \eta)$$
7.5 Linear and Shift-invariant Operators

Suppose we have a linear and shift-invariant operator acting on \( \phi(t) \) as follows:

\[
\phi_{\text{out}}(t) = \mathcal{X}_{\text{LSI}} \{ \phi(\cdot)(t) \} = \mathcal{X}_{\text{LSI}} \left\{ \int \phi(\xi) \delta(t-\xi) \, d\xi \right\}
\]

We have then

\[
\mathcal{X}_{\text{LSI}} \{ \phi(\cdot)(t) \} = \mathcal{X}_{\text{LSI}} \left\{ \int \phi(\xi) \mathcal{X}_{\text{LSI}} \{ \delta(t-\xi) \} \, d\xi \right\} = 
\]

by linearity

\[
= \int \phi(\xi)\mathcal{X}_{\text{LSI}} \{ \mathcal{X}_{\text{LSI}} \{ \delta(t-\xi) \} \} \, d\xi =
\]

by shift-invariance

\[
= \int \phi(\xi)\mathcal{X}_{\text{LSI}} \{ \mathcal{X}_{\text{LSI}} \{ \delta(t-\xi) \} \} \, d\xi =
\]

\[
= \int \phi(\xi)\mathcal{X}_{\text{LSI}} \{ \mathcal{X}_{\text{LSI}} \{ \delta(t-\xi) \} \} \, d\xi =
\]

Denote by:

\[
h_{\text{LSI}}(t) = \mathcal{X}_{\text{LSI}} \{ \delta(t) \}
\]

a function called the impulse response of \( \mathcal{X}_{\text{LSI}} \).
With this definition we have

\[ \text{out}(t) = \int_{-\infty}^{\infty} \phi(s) T_{s} \{ h_{L1}(t) \} \, ds = \int_{-\infty}^{t} \phi(s) h(s-t) \, ds \]

Hence the output is the convolution of the input with the (time) impulse response of the operator/system \( L_{1} \).

This is a general result of utmost importance. Any linear and shift invariant operator is completely described/defined by its impulse response, and the output of such an operator acting on an input is the convolution of the input with the system/operator's impulse response function.
Note that if we do not have shift invariance we can define, for a linear operator,
\[ X_1 \{ \delta(t - \xi) \} = h_1(t, \xi) \]
a set of functions parameterized by \( \xi \) which are the \( t=\xi \) impulse responses of \( X_1 \cdot \delta \)
(and in this case they are not \( \xi \)-shifted versions of \( h_2(t, 0) \) as before) and we have here

\[
\Phi(t) = \int_{-\infty}^{+\infty} \delta(t - \xi) h_2(t, \xi) d\xi
\]

(Hence we conclude that Example 3 is in fact the general description of any linear and linear+shift invariant operator.)
Some important properties of LSI-operators,
(i.e. of the convolution of two functions).
We have that
\[ \int_{-\infty}^{+\infty} f(\xi) g(t-\xi) d\xi = \int_{-\infty}^{+\infty} g(\xi') f(t-\xi') d\xi' \]

Hence we can write \( f \otimes g(t) = g(t) \otimes f(t) \)
where \( \otimes \) stands for the convolution.
We also have \( (f \otimes g) \otimes h = f \otimes (g \otimes h) \)
\( (f \otimes g) = f \)

This shows that Linear Shift Invariant operators commute, i.e.
\[ H^2_\text{LSI} \{ H^1_\text{LSI} \{ f(t) \} \} = (f(t) \otimes h^1_{\text{LSI}}(t)) \otimes h^2_{\text{LSI}}(t) = \]
\[ = (f(t) \otimes (h^1_{\text{LSI}}(t) \otimes h^2_{\text{LSI}}(t))) = (f(t) \otimes (h^2_{\text{LSI}}(t) \otimes h^1_{\text{LSI}}(t))) = \]
\[ = (f(t) \otimes h^2_{\text{LSI}}(t)) \otimes h^1_{\text{LSI}}(t) = \]
\[ = H^2_\text{LSI} \{ H^1_\text{LSI} \{ f(t) \} \} = \]