ON REPRESENTATIONS "WITH CHOICES"

(or How to find more than one "representative" number for a group of objects and how to use such sets of representatives)

2.1 Again consider a list of numbers, as before, ordered in nondecreasing order

\[ x_1 \leq x_2 \leq x_3 \ldots \leq x_n \quad x_i \in \mathbb{R} \]

How would we choose two or more numbers (say \( k \ll n \)) to represent them well? Let us denote the describing \( \theta \)'s by \( \theta_1, \theta_2, \ldots, \theta_k \). Clearly we shall be able to order them as follows \( \theta_1 < \theta_2 < \theta_k \).
In this case we shall have to map each $x_i$ to a corresponding $\theta_j$, as follows:

$$x_{v1}, x_2, \ldots, x_{v_1}, x_{v_1+1}, \ldots, x_{v_2}, \ldots, x_{v_k}, \ldots, x_N$$

$$\theta_1, \theta_1, \theta_1, \theta_2, \theta_2, \ldots, \theta_{k-1}, \theta_k.$$

We note that we map a monotonically increasing sequence of $x_i$'s to a strictly increasing sequence of $\theta_i$ by selecting the break-points $r_1, r_2, \ldots, r_{k-1}$.

To determine $\theta_1, \theta_2, \ldots, \theta_k$ we need the $r_j$'s and to determine $r_j$'s we need to rely on the data sequence and the optimization criterion. Let us invoke the MSE loss function. Then, for a sequence $\{r_j\}$ and numbers $\{\theta_j\}$ we can write:

$$\text{MSE} = \frac{1}{N} \left( \sum_{i=1}^{r_1} (x_i - \theta_1)^2 + \sum_{i=r_1+1}^{r_2} (x_i - \theta_{1+1})^2 + \sum_{i=r_{k-1}+1}^{N} (x_i - \theta_k)^2 \right)$$

$$= \frac{1}{N} \sum_{l=0}^{k-1} \sum_{i=r_l+1}^{r_{l+1}} (x_i - \theta_{l+1})^2$$

where $\{r_0, r_1, \ldots, r_k\} = \{0, N\}$. 

\[ r_j \neq 0 \]
and, given the data \( x_1, x_2, \ldots, x_N \), we want to determine \( \theta_j \) and \( \theta_j's \) to minimize this MSE loss function.

Clearly, given \( r_0 \) and \( r_{e+1} \), we have that

\[
\Theta_{e+1}^{(\text{optimal})} = \frac{1}{r_{e+1} - r_e} \sum_{i=r_e+1}^{r_{e+1}} x_i = x_i's \text{ averaged over the interval } [r_e+1, r_{e+1}]
\]

by our previous analysis.

We also know that:

\[
\sum_{i=r_e+1}^{r_{e+1}} (x_i - \Theta_{e+1}^{(\text{optimal})})^2 = \sum_{i=r_e+1}^{r_{e+1}} x_i^2 - \left( \frac{1}{r_{e+1} - r_e} \sum_{i=r_e+1}^{r_{e+1}} x_i \right)^2
\]

and this leads to an expression for \( \Psi_{\text{MSE}} \), dependent on the decision levels, or breakpoints \( r_1, r_2, \ldots, r_{e-1} \) only:

\[
\Psi_{\text{MSE}}(\ell, r_j) = \frac{1}{N} \left( \sum_{l=0}^{k-1} \sum_{i=r_{l+1}}^{r_{l+1}} x_i^2 - \sum_{l=0}^{k-1} \left( \frac{1}{r_{l+1} - r_l} \sum_{i=r_{l+1}}^{r_{l+1}} x_i \right)^2 \right)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} x_i^2 - \frac{1}{N} \sum_{l=0}^{k-1} \left( \frac{1}{r_{l+1} - r_l} \sum_{i=r_{l+1}}^{r_{l+1}} x_i \right)^2
\]
where we defined

\[ S(m) = \sum_{i=1}^{m} x_i \]

hence \( S(N) = \sum_{i=1}^{N} x_i \) and

\[ S(r_{l+1}) - S(r_e) = \sum_{i=r_e+1}^{r_{l+1}} x_i \] as needed.

Now in order to minimize \( \sum_{r_e}^{r_{l+1}} \) we have to maximize

\[ \frac{1}{N} \sum_{l=0}^{k-1} \frac{1}{(r_{l+1} - r_e)^2} \left[ S(r_{l+1}) - S(r_e) \right]^2 \]

Let us assume that all \( x_i \)'s are positive

\( \text{without loss of generality: since we can add} \)

\( \text{to every} \ x_i \ \text{a positive number} \ A \ so \ that} \)

\( A + x_1 \leq A + x_2 \leq A + x_3 \cdots \leq A + x_N, \)

and then \( \tilde{S}(m) = \sum_i x_i + mA \)

and we have \( \tilde{S}(r_{l+1}) - \tilde{S}(r_e) = \sum_{i=r_e+1}^{r_{l+1}} x_i + (r_{l+1} - r_e)A \)
Now

\[
\frac{1}{N} \sum_{l=0}^{k-1} \frac{1}{r_{e+1} - r_e} \left[ \tilde{S}(r_{e+1}) - \tilde{S}(r_e) \right]^2 =
\]

\[
= \frac{1}{N} \sum_{l=0}^{k-1} \frac{1}{r_{e+1} - r_e} \left[ \sum_{i=r_{e+1}}^{r_{e+1}} x_i \right]^2 + \sum_{l=0}^{b-1} \frac{2}{r_{e+1} - r_e} \sum_{i=r_{e+1}}^{r_{e+1}} x_i \left( r_{e+1} - r_e \right) A
\]

\[
+ \sum_{l=0}^{k+1} \frac{1}{r_{e+1} - r_e} A^2 \left( r_{e+1} - r_e \right)^2
\]

\[
= \frac{1}{N} \sum_{l=0}^{k+1} \frac{1}{r_{e+1} - r_e} \left( \sum_{i=r_{e+1}}^{r_{e+1}} x_i \right)^2 + 2A \sum_{i=1}^{N+1} x_i + A^2 \sum_{l=0}^{b-1} \left( r_{e+1} - r_e \right)
\]

\[
= \frac{1}{N} \left( \text{SAME} \right) + 2A \sum_{i=1}^{N+1} x_i + A^2 \cdot N
\]
Note also that
\[
\frac{1}{N} \sum_{i=1}^{N} (x_i + A)^2 = \frac{1}{N} \sum_{i=1}^{N} x_i^2 + 2A \frac{1}{N} \sum_{i=1}^{N} x_i + A^2 \frac{1}{N} \sum_{i=1}^{N} 1
\]

Hence we obtain that:
\[
\text{MSE}(\{r_i^j\}) \text{ for } A + x_1, A + x_2, \ldots, A + x_N
\]
is identical to that for \(x_1, x_2, \ldots, x_N\)!

\[
= \frac{1}{N} \sum_{i=1}^{N} x_i^2 - \frac{1}{N} \sum_{k=0}^{k-1} \frac{1}{(c_k - c_{k-1})^2} \left[ S(c_{k+1}) - S(c_k) \right]^2
\]

Therefore we have that
\[
S(m) \text{ is a strictly increasing function of } m.
\]
Now we have the following problem:

- given a strictly increasing function of \( m \), \( S(m) \) that obeys, \( S(0) = 0 \)

\[
\frac{S(m+1) - S(m)}{x_{m+1}} \geq \frac{S(m) - S(m-1)}{x_m}
\]

- maximize w.r.t. the breakpoints

\[
r_0 = 0 < r_1 < r_2 < \ldots < r_{k-1} < r_k = N
\]

the expression

\[
\sum_{l=0}^{k-1} \frac{1}{(r_{l+1} - r_l)} \left[ S'(r_{l+1}) - S'(r_l) \right]^2
\]

s.t.

\[
\sum_{l=0}^{k-1} (S(r_{l+1}) - S(r_l)) = S(N) \text{- given}
\]

\[
\sum_{l=0}^{k-1} \frac{1}{(x_{l+1} - x_l)^2} \sum \alpha_i^2 (x_{l+1} - x_l) = S(N)
\]
Let us rewrite the optimization problem as follows:

$$\max \sum_{l=0}^{k-1} \left( \frac{S(r_{e+1}) - S(r_e)}{r_{e+1} - r_e} \right) (r_{e+1} - r_e)$$

subject to

$$\sum_{l=0}^{k+1} \left( \frac{S(r_{e+1}) - S(r_e)}{r_{e+1} - r_e} \right) (r_{e+1} - r_e) = S'(N)$$

given

$S'(m)$ is given for $m=0,1,2,\ldots,N$

Note that we have that the function $S$ is by definition a convex function and $S'(m) \triangleq \frac{S(m+\delta) - S(m)}{\delta}$ is monotonically nondecreasing.
It is difficult to proceed further in the optimization process to determine the best breakpoints as integers. Hence we proceed as follows.

Let us regard the real line with the numbers $x_1, x_2, \ldots, x_n$ as the range in which a random variable $X_n$ takes values and assume its realizations are $x_i$ with probability $1/N$ for each $i$. Hence the probability distribution function is formally

$$p(x) = \sum_{i=1}^{N} \frac{1}{N} \delta(x-x_i)$$

where $\delta(.)$ is the impulse or Dirac function with properties

$$\begin{cases} 
\delta(x) = 0 \text{ for all } x \neq 0 \\
\int_{-\infty}^{\infty} \delta(x) dx = 1
\end{cases}$$
We clearly have here:

\[
E \{ X_\omega \} = \int_{-\infty}^{+\infty} x \, p(x) \, dx = \sum_{i=1}^{N} x_i \cdot \frac{1}{N} = \frac{1}{N} \sum_{i=1}^{N} x_i
\]

Now we shall define a way to represent the realizations of the random variable \( X_\omega \) by a set of \( K \) representation values \( \theta_1, \theta_2, \ldots, \theta_K \) as follows:

Select threshold levels \( r_1, r_2, \ldots, r_{K-1} \) and define a mapping function \( Q(x) \) as follows \( Q : \mathbb{R} \to \{ \theta_1, \theta_2, \ldots, \theta_K \} \) \((\theta_1 < \theta_2 < \theta_2 < \ldots < \theta_K)\):

\[
Q(x) = \begin{cases} 
\theta_i & \text{if } r_{i-1} < x < r_i \\
\theta_{i+1} & \text{if } x \leq r_{i-1} \\
\theta_K & \text{if } x \geq r_{K-1} 
\end{cases}
\]

This function will be called a K-level representation or "quantization" function.
2.3 With the help of these definitions we shall restate our problem of representing the data $x_1, x_2, \ldots, x_n$ as one of representing the random variable $X_\omega$ with $K$ values $\theta_1, \theta_2, \ldots, \theta_K$, "optimally" by suitable choices of $r_1, r_2, \ldots, r_{k-1}$ and $\theta_1, \theta_2, \ldots, \theta_K$.

The optimality criterion will be the (expected, w.r.t. the distribution $p(x)$) mean squared error, i.e.

$$\text{ESE} = \int_{-\infty}^{+\infty} (x - Q(x))^2 p(x) dx = \sum_{i=1}^{K} \int_{r_{i-1}}^{r_i} (x - \theta_i)^2 p(x) dx$$

where we define $r_0 = -\infty$ and $r_K = +\infty$ (the endpoints of the range of $X_\omega$). Note that we managed to map our original, discrete
problem into a problem with "continuous \((R)\)"
variables \(\Theta_1, \Theta_2, \ldots, \Theta_N\) and \(r_1, r_2, \ldots, r_{k-1}\).

First note that if we set the values \(r_1, r_2, \ldots, r_{k-1}\), then the optimal choices for \(\Theta_1, \Theta_2, \ldots, \Theta_N\) are completely determined, since clearly we have

\[
\min_{\Theta} \int_{r_{i-1}}^{r_i} (x - \Theta)^2 p(x) \, dx =
\]

\[
\min_{\Theta} \int_{r_{i-1}}^{r_i} (x^2 - 2\Theta x + \Theta^2) p(x) \, dx \Rightarrow
\]

\[
\Theta_{i \text{ optimal}} = \frac{1}{\int_{r_{i-1}}^{r_i} p(x) \, dx} \int_{r_{i-1}}^{r_i} x p(x) \, dx
\]

\[
= \frac{1}{\sum_{\text{of } x_i} \sum_{r_{i-1} < x_i < r_i} \frac{1}{N}} \sum_{i} x_i \frac{1}{N}
\]

as before!
An important observation that we did not see immediately in the discrete case is that, if we set the representation values $\Theta_1, \Theta_2, \Theta_3 \ldots \Theta_n$ and we have a value $x \in \mathbb{R}$ as a realization of $x_{\omega}$, to minimize $\Psi_{E-MSE}$ we must assign $x$ to the closest representation value i.e. we must ask which number $|x - \Theta_i|$ is smallest.

Hence, the decision levels $r_1, r_2, \ldots r_{k-1}$ are determined by the levels $\Theta_1, \Theta_2, \ldots \Theta_n$ via:

$$r_i = \frac{\Theta_i + \Theta_{i+1}}{2} \quad i = 1, 2, \ldots k-1$$

i.e. the midpoints of the intervals between the representation levels, irrespective of the distribution of $x$'s!
Therefore if we know the thresholds $r_1, r_2, \ldots, r_{K-1}$ we can determine the representation levels and if we know the representation levels $\theta_1, \theta_2, \ldots, \theta_K$ we can determine the thresholds. However we have to determine both of these, jointly, optimizing for the function $Q(x)$ w.r.t. the criterion $\Psi_{ESE}$.

$$\Psi_{ESE}^{(r_{1, \ldots, K-1}, \theta_1, \theta_2, \ldots, \theta_K)} = \sum_{i=1}^{K} \int_{r_{i-1}}^{r_i} (x - \theta_i)^2 p(x) \, dx$$

To minimize this function w.r.t. the $2K-1$ variables we write all the necessary conditions for a minimum (in fact for a stationary point!)
\[ \frac{d\psi_{\text{ESE}}}{d\theta_i} = 0 \quad \text{and} \quad \frac{d\psi_{\text{ESE}}}{dr_i} = 0 \]

and this yields:

\[
\begin{align*}
\int_{r_{i-1}}^{r_i} (x - \theta_i) p(x) \, dx &= 0 \quad i = 1, 2, \ldots, k \\
(r_i - \theta_i)^2 p(r_i) - (r_i - \theta_{i+1})^2 p(r_i) &= 0 \quad i = 1, 2, \ldots, k-1
\end{align*}
\]

Assuming that \( p(r_i) \neq 0 \) (which in our case is not true at all points, but let us forget this!) we get:

\[
\begin{align*}
\theta_i &= \frac{1}{\int_{r_{i-1}}^{r_i} p(x) \, dx} \int_{r_{i-1}}^{r_i} x p(x) \, dx \\
\int_{r_{i-1}}^{r_i} p(x) \, dx &= \theta_i + \theta_{i+1} \\
r_i &= \frac{\theta_i + \theta_{i+1}}{2}
\end{align*}
\]

The conditions we have seen before! These are necessary conditions for optimality.
2.4 Suppose we managed to get the solution of the optimization problem: what will be the expected mean squared error? Well, we shall have

\[ \text{ESE}_{(\text{opt})} = \sum_{i=1}^{k} \int_{r_{i-1}}^{r_i} \frac{(x-\theta_i)^2}{\left( x^2 - 2\theta_i x + \theta_i^2 \right) p(x)} \, dx = \]

\[ = \int_{-\infty}^{+\infty} x^2 p(x) \, dx - 2 \sum_{i=1}^{k} \theta_i \int_{r_{i-1}}^{r_i} x \, p(x) \, dx + \]

\[ \sum_{i=1}^{k} \theta_i^2 \int_{r_{i-1}}^{r_i} p(x) \, dx = \]

\[ = \int_{-\infty}^{+\infty} x^2 p(x) \, dx - \sum_{i=1}^{k} \theta_i^2 \int_{r_{i-1}}^{r_i} p(x) \, dx = \]

\[ = \int_{-\infty}^{+\infty} x^2 p(x) \, dx - \sum_{i=1}^{k} \frac{1}{\int_{r_{i-1}}^{r_i} p(x) \, dx} \left( \int_{r_{i-1}}^{r_i} x p(x) \, dx \right)^2. \]

\[ = \int_{-\infty}^{+\infty} x^2 p(x) \, dx - \sum_{i=1}^{k} \theta_i^2 \text{Pr} \{ r_{i-1} < x < r_i \} \]
How should we determine the optimal representation levels $\Theta_1, \Theta_2, \ldots \Theta_k$ (and the corresponding decision levels between them $r_i$, for $i = 1, 2, \ldots K-1$).

Well, here comes a wonderful idea:

Start with some "arbitrary" decision levels \{r_1^0, r_2^0, \ldots, r_{K-1}^0\} (say uniformly spaced between $x_1$ and $x_N$). Compute the corresponding optimal representations over the regions defined by \{r_1^0, \ldots, r_{K-1}^0\}. And get \{\Theta_1^0, \Theta_2^0, \ldots, \Theta_k^0\}. Now having these levels we can compute new decision levels

\[
\{ r_i^{(1)} = \frac{\Theta_i^0 + \Theta_{i+1}^0}{2} \text{ for } i = 1, \ldots K-1 \}
\]

and proceed with these as before!

How do we know that we improve?
Because, at each step we optimize, i.e. reduce the \( \bar{\gamma}_{ESE} \), hence we certainly improve. This process is called a coordinate descent process. It is known as the Lloyd-Max algorithm!

The process decreases the expected mean squared error at every step, i.e.

\[
\bar{\gamma}_{ESE}^t \leq \bar{\gamma}_{ESE}^{t+1}
\]

and we can stop it when the improvement becomes negligible.

(HW): Let us see an example of this process. Assume that \( X_w \) is a random variable with uniform distribution over \([x_L, x_H]\) or \([0, 1]\) w.l.o.g. Hence the pdf \( p(x) \)

over \([0, 1]\) is

\[
p(x) = \begin{cases} 
0 & x \in [0, 1] \\
1 & x \in [0, 1]
\end{cases}
\]
Let us start a Lloyd-Max process with arbitrary decision levels $r_1, r_2, \ldots, r_k \in (0, 1)$. We have for the given pdf $p(x)$ that

$$\Theta_i = \frac{1}{r_i - r_{i-1}} \int_{r_{i-1}}^{r_i} x \, dx =$$

$$= \frac{1}{r_i - r_{i-1}} \left( \frac{x^2}{2} \right)_{r_{i-1}}^{r_i} = \frac{1}{2} \left( \frac{r_i^2 - r_{i-1}^2}{r_i - r_{i-1}} \right) = \frac{r_{i-1} + r_i}{2}$$

and

$$r_i = \frac{\Theta_i + \Theta_{i+1}}{2}$$

Hence given $r_1^0, r_2^0, \ldots, r_k^0$ with $r_0^0 = 0, r_k^0 = 1$, we get

$$\Theta_1^0 = \frac{r_1^0 + r_2^0}{2}, \quad \Theta_2^0 = \frac{r_2^0 + r_3^0}{2}, \ldots, \quad \Theta_k^0 = \frac{r_{k-1}^0 + r_k^0}{2}$$

and then

$$r_1' = \frac{r_1^0 + r_2^0 + r_3^0}{2}, \quad r_2' = \frac{r_2^0 + r_3^0 + r_4^0}{2}, \ldots \text{ etc}$$

i.e.

$$r_1' = \frac{r_1^0}{4} + \frac{r_2^0}{4} + \frac{r_3^0}{4}, \quad r_2' = \frac{r_1^0}{2} + \frac{r_2^0}{2} + \frac{r_3^0}{2}, \ldots \text{ etc}$$
and in general:

\[ r_i^{\text{next}} = \frac{r_{i-1} + r_i + r_i + r_{i+1}}{2} = \frac{r_{i-1}}{4} + \frac{r_i}{2} + \frac{r_{i+1}}{4} \]

or the matrix iteration:

\[
\begin{bmatrix}
    r_0 \\
    r_1 \\
    r_2 \\
    \vdots \\
    r_{k-1} \\
    \hline
    r_k
\end{bmatrix}^{\text{next}} =
\begin{bmatrix}
    1 & 0 & 0 \\
    \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
    \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
    \vdots \\
    \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
    0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    r_0 \\
    r_1 \\
    r_2 \\
    \vdots \\
    r_{k-1} \\
    \hline
    r_k
\end{bmatrix}^{\text{pre}}
\]

or

\[
\begin{bmatrix}
    r_1 \\
    r_2 \\
    \vdots \\
    r_{k-1} \\
    \hline
    r_k
\end{bmatrix}
= \begin{bmatrix}
    \frac{1}{2} & \frac{1}{4} \\
    \frac{1}{4} & \frac{1}{4} \\
    \vdots \\
    \frac{1}{4} & \frac{1}{4} \\
    \frac{1}{4} & \frac{1}{4}
\end{bmatrix}
\begin{bmatrix}
    r_1 \\
    r_2 \\
    \vdots \\
    r_{k-1} \\
    \hline
    r_k
\end{bmatrix}
\]

A matrix \( M \) has the following eigenstructure:

\[ M = R \Lambda L^T \]

where the eigenvalues are \( \lambda_0 \) and \( \lambda_k = 1 \) and all others are given by

\[ \lambda_j = 1 - \frac{1}{2} (1 - \cos j \theta) \quad \text{for} \quad \theta = \frac{\pi}{k} \]

hence \( |\lambda_j| < 1 \) for all \( j \).
and the eigenvectors (right and left) are biorthogonal

\[ R = \begin{bmatrix}
\frac{2}{k-1} & 0 & \cdots & 0 \\
0 & \sin j \theta & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & \sin (k-1) \theta \\
\frac{2}{k-1} & 0 & \cdots & 0 \\
\frac{2}{k-1} & 0 & \cdots & 0 \\
\frac{2}{k-1} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & \frac{2}{k-1}
\end{bmatrix}
\]

\[ L = \begin{bmatrix}
0 & -\frac{1}{2} \sin \frac{\theta}{2} & \cdots & 0 \\
0 & \sin j \theta & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & \sin (k-1) \theta \\
\frac{k-1}{2} & 0 & \cdots & 0
\end{bmatrix}
\]

with \( L_i \cdot R_j = \delta_{ij} = \{ 1 \text{ if } i = j, \ 0 \text{ if } i \neq j \}
\]

This shows that \( R^T L^T \)

\[ M = R^T L^T \Rightarrow \]

\[ \bar{F}(m) = R_0 \langle L_0, F(0) \rangle + R_k \langle L_k, F(0) \rangle + \text{small terms} \]

\[ \Rightarrow \bar{F}(\infty) = \left( r_0(0), \frac{k_0+1}{k}, \frac{2}{k}, \cdots, \frac{k-1}{k}, \frac{1}{k}, \frac{r_0(1)}{k} \right) \]
Hence for a uniform distribution of $X_w$ the best decision levels to which the Lloyd Max algorithm converges are uniformly spread in the interval $[0, 15]$ i.e.

$$r_i = i \frac{1}{K} \quad \text{for} \quad k = 0, 1, 2, \ldots, k$$

In this case the expected (mean) squared error will be

$$\Psi_{ESE \ (opt)} = \int_{X_L}^{X_H} \frac{x^2}{x_{i+1} - x_i} \, dx - \sum_{i=1}^{k} \left( \frac{r_{i+1} - r_i}{2} \right)^2 \frac{1}{k}$$

$$= \frac{1}{3} \left( \frac{X_H^3 - X_L^3}{X_H - X_L} \right) - \frac{1}{k} \sum_{i=1}^{k} \left( \frac{i-1+i}{2k} \right)^2 \left( x_{i+1} - x_i \right)^2$$

$$= \frac{(X_H - X_L)^2}{3} - \frac{1}{k} \left( \frac{X_H - X_L)^2}{4k^2} \sum_{i=1}^{k} (2i-1)^2 \right)$$

$$= \frac{(X_H - X_L)^2}{3} - \frac{1}{k} \left( \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{4k^2-1}{k^3} \right) \frac{k(2k-1)(2k+1)}{k^3}$$

$$= \frac{(X_H - X_L)^2}{12} \left( \frac{1}{4} - \frac{1}{3} \cdot \frac{4k^2-1}{k^3} \right) = \frac{(X_H - X_L)^2}{12k^2}$$
Note that from the optimality conditions for the uniform distribution, we could seek to find solutions for the 2K-1 equations with 2K-1 unknowns, i.e. $\Theta_1, \Theta_2, \ldots, \Theta_K, r_1, r_2, \ldots, r_{K-1}$ with $r_0 = 0$ and $r_K = 1$, as follows:

\[
\begin{align*}
\Theta_1 &= \frac{1}{2}r_0 + \frac{1}{2}r_1 \\
\Theta_2 &= \frac{1}{2}r_1 + \frac{1}{2}r_2 \\
\vdots \\
\Theta_K &= \frac{1}{2}r_{K-1} + r_1 \\
r_1 &= \frac{1}{2}\Theta_1 + \frac{1}{2}\Theta_2 \\
\vdots \\
r_{K-1} &= \frac{1}{2}\Theta_{K-1} + \Theta_K \\
r_0 &= 0 \\
r_K &= 1
\end{align*}
\]

and realize that the equation

\[
\begin{bmatrix}
1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} & \frac{1}{2} \\
0 & -1 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & -1 & \frac{1}{2} & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & -1 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
\Theta_1 \\
\Theta_2 \\
\vdots \\
\Theta_K \\
r_1 \\
r_2 \\
\vdots \\
r_K
\end{bmatrix}
= \begin{bmatrix}
\left(\frac{1}{K} + \frac{1}{K}\right) & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\left(\frac{1}{K} + \frac{1}{K}\right) & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\left(\frac{1}{K} + \frac{1}{K}\right) & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\left(\frac{1}{K} + \frac{1}{K}\right) & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\left(\frac{1}{K} + \frac{1}{K}\right) & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\left(\frac{1}{K} + \frac{1}{K}\right) & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
\left(\frac{1}{K} + \frac{1}{K}\right) & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\end{bmatrix}
\]

is solved uniquely by $\{r_i = \frac{i}{K}, i = 0, 1, \ldots, K, \Theta_j = \left[\frac{j}{K} + \frac{j}{K} - \frac{1}{K} - \frac{1}{K}\right] = 1, \ldots\}$.
Note that by using the best mapping function $Q(\cdot)$, we replace a random variable $X_w$ with another, defined by

$$
\Theta_w \triangleq Q(X_w) \in [\theta_1, \theta_2, \ldots, \theta_k]
$$

that takes values in a finite set of representation levels. The error random variable is

$$
\varepsilon_w \triangleq X_w - \Theta_w
$$

and we have that

$$
E(\varepsilon_w^2) = \gamma_{ES\beta}^{(\text{optimal})} =
$$

$$
= \int x^2 p(x) \, dx - \sum_{i=1}^{k} \Theta_i^2 \Pr\{\Theta_w = \Theta_i\} =
$$

$$
= E(X_w^2) - E(\Theta_w^2)
$$

Hence

$$
E(\varepsilon_w^2) = E((X_w - \Theta_w)^2) =
$$

$$
= E(X_w^2 - 2X_w\Theta_w + \Theta_w^2) =
$$

$$
= E(X_w^2) - E(\Theta_w^2) \text{ yielding}
$$
\[ E(x_w^2) - 2E(x_w \theta_w) + E(\theta_w^2) = E(x_w^2) - E(\theta_w) \]

or

\[ 2E(x_w \theta_w) - 2E(\theta_w^2) = 0 \]

or

\[ E(x_w - \theta_w) \theta_w = 0 \]

This shows that, while \( \theta_w \) is clearly strongly dependent on \( x_w \) (in fact \( \theta_w \leq Q(x_w) \)), we see that the error \( \varepsilon_w \) is "orthogonal" to \( \theta_w \) in the sense that

\[ E_w(\varepsilon_w \theta_w) = 0 \]

So \( \theta_w \) is the "projection" of the random variable \( x_w \) on the space of random variables defined by \( Q(\varepsilon_w \theta_w \mid \varepsilon_w \text{ drawn according to } x_w) \)
2.7 "Approximately optimal quantization."

Suppose that the number of quantization levels is large and the range of $X$ is finite $[x_L, x_H]$, and we have $p_X(x)$ over $[x_L, x_H]$. If $K$ is large we shall divide the interval $[x_L, x_H]$ into many small intervals via the decision levels $r_0 = x_L, r_1, r_2, \ldots, r_{K-1}, r_K = x_H$. In this case the expected squared error for $Q(.)$, designed with these levels and the representation levels $\Theta_1, \Theta_2, \ldots, \Theta_K$ is given by

$$ESE(Q) = \sqrt{\int_{x_L}^{x_H} (x - Q(x))^2 p(x) dx} =$$

$$= \sum_{i=1}^{K} \int_{r_{i-1}}^{r_i} (x - \Theta_i)^2 p(x) dx =$$
\[
\sum_{i=1}^{k} \int_{r_{i-1}}^{r_i} (x-\theta_i)^2 p\left(\frac{r_{i-1}+r_i}{2}\right) dx = 0
\]

\[
= \sum_{i=1}^{k} p\left(\frac{r_{i-1}+r_i}{2}\right) \frac{(x-\theta_i)^3}{3} \bigg|_{r_{i-1}}^{r_i} = \frac{1}{3} \sum_{i=1}^{k} p\left(\frac{r_{i-1}+r_i}{2}\right) \left[ (r_i-\theta_i)^3 - (r_{i-1}-\theta_i)^3 \right]
\]

Here we can determine the optimal \(\theta_i\)'s and \(r_i\)'s as follows:

\[
\frac{\partial \Psi_{\text{esse}}}{\partial \theta_i} = 0 \Rightarrow \frac{1}{3} p\left(\frac{r_{i-1}+r_i}{2}\right) \left[ -3(r_i-\theta_i)^2 + 3(r_{i-1}-\theta_i)^2 \right]
\]

\[
= p\left(\frac{r_{i-1}+r_i}{2}\right) \left[ (r_{i-1}-\theta_i + r_i - \theta_i)(r_{i-1}-\theta_i - r_i + \theta_i) \right]
\]

\[
= p\left(\frac{r_{i-1}+r_i}{2}\right) \left( r_{i-1} + r_i - 2\theta_i \right) \left( r_{i-1} - r_i \right) = 0
\]

This yields

\[
\theta_i^{\text{opt}} = \frac{r_{i-1} + r_i}{2}
\]
Setting \( \Theta_i^* \) in the expression for the expected squared error, we obtain:

\[
\sqrt{\text{ESE}}(\Theta_1^*, \Theta_2^*, \ldots, \Theta_k^*) =
\]

\[
= \frac{1}{3} \sum_{i=1}^{k} p\left( \frac{r_{i-1} + r_i}{2} \right) \left[ \left( \frac{r_i - r_{i-1}}{2} \right)^3 - \left( \frac{r_{i+1} + r_i}{2} \right)^3 \right]
\]

\[
= \frac{1}{3} \sum_{i=1}^{k} p\left( \frac{r_{i-1} + r_i}{2} \right) \left[ \left( \frac{r_i - r_{i-1}}{2} \right)^3 - \left( \frac{r_i - r_{i+1}}{2} \right)^3 \right] =
\]

\[
= \frac{1}{3} \sum_{i=1}^{k} p\left( \frac{r_{i-1} + r_i}{2} \right) \left[ \frac{2 (r_i - r_{i-1})^3}{8} \right] =
\]

\[
= \frac{1}{12} \sum_{i=1}^{k} p\left( \frac{r_{i-1} + r_i}{2} \right) (r_i - r_{i-1})^3
\]

Where note that \( r_0 \equiv X_L \) and \( r_k \equiv X_H \).
This yields:

\[
\Psi_{\text{ESE}} (x_1, x_2, \ldots, x_k, y) = \frac{1}{12} \sum_{i=1}^{k} \left( \frac{1}{3} \left( \frac{r_i r_{i+1}}{2} \right) \right) (r_i - r_{i-1})^3
\]

\[
= \frac{1}{12} \sum_{i=1}^{k} \mu_i^3 \quad \text{where} \quad \mu_i = \frac{1}{3} \left( \frac{r_i r_{i+1}}{2} \right) (r_i - r_{i-1})
\]

Note that we have

\[
\sum_{i=1}^{k} \mu_i = \sum_{i=1}^{k} \frac{1}{3} \left( \frac{r_i r_{i+1}}{2} \right) (r_i - r_{i-1}) = \int_{x_1}^{x_2} \left( c(q) \right)^{1/3} dq \quad \text{a constant}
\]

So we are led to determine the decision levels, via \( \mu_i \) by minimizing:

\[
\Psi_{\text{ESE}} = \frac{1}{12} \sum_{i=1}^{k} \mu_i^3
\]

Subject to \( \sum_{i=1}^{k} \mu_i = \text{a constant} (M) \).
Using Lagrange multipliers we obtain

\[
\text{minimize } \left( \frac{1}{12} \sum_{i=1}^{k} \mu_i^3 + \lambda (M - \sum_{i=1}^{k} \mu_i) \right)
\]

or

\[
\frac{\partial}{\partial \mu_i} \left( \frac{1}{12} \sum_{i=1}^{k} \mu_i^3 + \lambda (M - \sum_{i=1}^{k} \mu_i) \right) = 0
\]

yielding

\[
\frac{1}{4} \mu_i^2 - \lambda = 0 \Rightarrow \mu_i = 2\sqrt{\lambda}
\]

To satisfy the constraint we need for all \(i\)’s

\[
\sum_{i=1}^{k} \mu_i = M \text{ or } 2\sqrt{\lambda} k = M
\]

which yields

\[
2\sqrt{\lambda} = \frac{M}{k}
\]

From here we obtain the optimal \(\mu_i\)’s as

\[
\mu_i^* = \frac{M}{k} = \text{constant}
\]
We obtained, therefore, the optimality conditions for the decision levels as follows:

\[ p^{\frac{1}{3}} \left( \frac{r_i - r_{i-1}}{2} \right) (r_i - r_{i-1}) = \frac{\int_{x_l}^{x_u} p(x)^{\frac{1}{3}} dx}{K} \]

Define a function

\[ \Delta(r) = \int_{x_l}^{r} p^{\frac{1}{3}}(x) dx \]

and look at the \( r_i \)'s for which

\[ \Delta(r_i) = \frac{i}{K} \int_{x_l}^{x_u} p^{\frac{1}{3}}(x) dx = \frac{i}{K} \Delta(x_u) \]

We have clearly

\[ \Delta(r_{i-1}) = \int_{x_l}^{r_{i-1}} p^{\frac{1}{3}}(x) dx = \frac{i-1}{K} \int_{x_l}^{x_u} p^{\frac{1}{3}}(x) dx \]

and

\[ \Delta(r_i) = \int_{x_l}^{r_i} p^{\frac{1}{3}}(x) dx = \frac{i}{K} \int_{x_l}^{x_u} p^{\frac{1}{3}}(x) dx \]
Therefore

\[ \Delta(r_i) - \Delta(r_{i-1}) = \frac{1}{k} \int_{x_L}^{x_U} p_{12}(x) dx \]

But \( \Delta(r_i) - \Delta(r_{i-1}) \leq \)

\[ \int_{r_{i-1}}^{r_i} p_{12}(x) dx = p_{12}(\frac{r_{i-1} + r_i}{2})(r_i - r_{i-1}) \]

Hence we see that the decision levels \( r_1, r_2, \ldots, r_{k-1} \) are the places where the function \( \Delta(r) \) crosses the levels \( i \cdot M/k \) or \( \int_{x_L}^{x_U} p_{12}(x) dx/k \) for \( i = 1, \ldots, k-1 \).
Hence the "approximately optimal" quantizer is

\[ X_w = x \]

\[ v = \Delta(x) \in [0, M] \]

The output is

\[ \Theta^* = \frac{v_i + v_{i-1}}{2} \]

This quantizer achieves the approximately optimal mean squared error of:

\[ \sqrt{\text{ESE}} \left( \text{optimal} \right) = \frac{1}{12} \sum_{i=1}^{K} (M_i^*)^3 = \frac{1}{12} K \cdot \frac{M^3}{K^3} = \frac{1}{12} \left( \frac{\int_{0}^{M} \frac{dy}{x} \, dx}{K^2} \right)^3 = \frac{1}{12} \frac{\text{Constant}}{K^2} \]

with \( \text{Constant} = M^3 \)
2.8. High-K uniform quantization

Suppose that the number of quantization levels is high and the range of values for a random variable \( X \) is \([x_L, x_U]\) and we have \( p(x) \) over the range.

Assume now that we simply divide the range \([x_L, x_U]\) into \( K \) intervals given by

\[
\Delta x_i = [x_L + (i-1) \frac{x_U - x_L}{K}, x_L + i \frac{x_U - x_L}{K}) \quad \text{for} \quad i = 1, 2, \ldots, K.
\]

It is clear that with this division the least mean squared error will be

\[
\text{ESE}(Q^u) = \int_{x_L}^{x_U} (x - Q^u(x))^2 p(x) dx =
\]

\[
= \frac{1}{2} \sum_{i=1}^{K} p \left( r_i + \frac{r_i}{2} \right) \int_{r_i}^{r_{i+1}} (x - \Theta_i)^2 dx =
\]

\[
= \frac{1}{2} \sum_{i=1}^{K} p \left( r_i + \frac{r_i}{2} \right) \left[ \frac{(r_i - \Theta_i)^3}{3} - \frac{(r_i - \Theta_{i-1})^3}{3} \right]
\]

and the best \( \Theta_i \)'s are, like before

\[
\Theta_i^{opt} = x_L + (i-1) \frac{x_U - x_L}{K} + \frac{1}{2} \frac{x_U - x_L}{K}
\]
i.e. the interval $\Delta_{xi}$'s midpoint. Setting these values of $\theta_i^{\text{opt}}$ in the expression for the expected squared error, one obtains

$$\Psi_{\text{ESE}}^{-n} = E(X^2_n - Q_n(X^2_n))^2 =$$

$$= \frac{1}{12} \sum_{i=1}^{K} p_x(\text{midpoint of } \Delta_i) \cdot \frac{(x^*_i - x_1)^3}{K^3} =$$

$$= \frac{1}{12} \frac{(x^*_i - x_1)^3}{K^2} \sum_{i=1}^{K} p_x(\text{midpoint of } \Delta_i) |\Delta_i| =$$

$$= \frac{1}{12} \frac{(x^*_i - x_1)^3}{K^2} \int_{x_1}^{x^*_i} p_x(x) \, dx =$$

$$= \frac{1}{12} \frac{(x^*_i - x_1)^2}{K^2}$$

Note that by optimizing the decision levels with a nonuniform quantizer adapted to $p_x(x)$ we obtained (for high $K$)

$$\Psi_{\text{ESE}} = \frac{1}{12} \frac{1}{K^2} \left[ \int_{x_1}^{x^*_i} p_x^{1/2}(x) \right]^2$$
This result begs the question: are we indeed doing better for the nonuniform "adapted" quantizer, i.e. do we have

\[ \Psi_{\text{ESE}}(Q^{\text{optimal}}) \leq \Psi_{\text{ESE}}(Q^{\text{uniform}}) \]

i.e.

\[ \frac{1}{12} \frac{1}{k^2} \sum_{x_L} p_x^{\frac{1}{3}} (x) \left[ \frac{1}{3} (x^4 - x^2) \right] \leq \frac{1}{12} \frac{1}{k^2} (x^4 - x_L^2) \]

The answer, is of course that we are doing better, due to the Hölder inequality, which states: that for two functions \( f, g \) we have

\[ \int f g dx \leq \left[ \int |f|^{p} dx \right]^\frac{1}{p} \left[ \int |g|^{q} dx \right]^\frac{1}{q} \]

for \( p, q \in [1, \infty) \) satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \) i.e. \( \frac{1}{2} = 1 - \frac{1}{p} = \frac{p-1}{p} \)

\[ q = \frac{p}{p-1} \]

Now take:

\[ f = p_{x^{\frac{1}{3}}(x)} \text{ and } g = 1 \text{ for } x \in [x_L, x_U] \]

we get:

\[ \int_{x_L}^{x_U} p_{x^{\frac{1}{3}}} \cdot 1 dx \leq \left[ \int |p_{x^{\frac{1}{3}}}^{3} dx \right]^\frac{1}{3} \left[ \int |x^{\frac{2}{3}} dx \right]^\frac{2}{3} \]
and this yields

\[ \left[ \int_{x_L}^{x_H} p_x^{1/3}(x) \, dx \right]^3 \leq \left[ \int_1^{x_H} p_x(x) \, dx \right] \cdot (x_H - x_L)^2 \]

or

\[ \left( \int_{x_L}^{x_H} p_x^{1/3}(x) \, dx \right)^3 \leq (x_H - x_L)^2 \]

as we expected (and equality happens only if there are real numbers \( \alpha, \beta > 0 \)

\[ \alpha \triangleq \int_{x_L}^{x_H} g(x) \, dx = (x_H - x_L) \quad (q = \frac{3}{2}) \]

\[ \beta \triangleq \int_{x_L}^{x_H} p(x) \, dx = \int_{x_L}^{x_H} p(x) \, dx - 1 \quad (p = 3) \]

So that

\[ (x_H - x_L) \cdot p_x(x) = 1 \quad \text{almost everywhere} \]

\[ \text{i.e.} \quad p_x(x) = \frac{1}{x_H - x_L} \quad \text{is the uniform distribution} \]

So we are always doing better vs the uniform quantizer except for the case when \( p_x(x) \) is uniform i.e. when we know that the uniform quantizer is best!