Tutorial 9
The Discrete Fourier Transform (DFT)

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Technion, CS Department
The DFT Matrix

The DFT matrix of size $M \times M$ is defined as

$$[\text{DFT}] = \frac{1}{\sqrt{M}} \begin{bmatrix} (W^*)^{0\cdot0} & \ldots & (W^*)^{0\cdot(M-1)} \\ \vdots & \ddots & \vdots \\ (W^*)^{(M-1)\cdot0} & \ldots & (W^*)^{(M-1)\cdot(M-1)} \end{bmatrix}$$

where $W = e^{i\frac{2\pi}{M}}$

and $^*$ denotes complex conjugate

The DFT matrix is symmetric and unitary, hence, its inverse is

$$[\text{DFT}]^* \quad = \quad \frac{1}{\sqrt{M}} \begin{bmatrix} W^{0\cdot0} & \ldots & W^{0\cdot(M-1)} \\ \vdots & \ddots & \vdots \\ W^{(M-1)\cdot0} & \ldots & W^{(M-1)\cdot(M-1)} \end{bmatrix}$$

i.e., $[\text{DFT}]^*[\text{DFT}] = [\text{DFT}][\text{DFT}]^* = I$
Representation of a Discrete Signal in the DFT Domain

Uniform sampling of the continuous signal $\varphi(t)$ provides us the discrete set of $M \triangleq 2N + 1$ samples:

$$\varphi_{-N}, \varphi_{-(N-1)}, \ldots, \varphi_0, \ldots, \varphi_{N-1}, \varphi_N$$

that can be also arranged in a column vector as follows:

$$\varphi = \begin{bmatrix}
\varphi_0 \\
\varphi_1 \\
\vdots \\
\varphi_N \\
\varphi_{-N} \\
\varphi_{-(N-1)} \\
\vdots \\
\varphi_{-1}
\end{bmatrix}$$

Note the position of the negative-indexed samples.
Representation of a Discrete Signal in the DFT Domain

The representation of the discrete signal \( \varphi \) is

\[
\varphi^F = [\text{DFT}] \varphi
\]

or in a more explicit form:

\[
\begin{bmatrix}
\varphi_0^F \\
\varphi_1^F \\
\vdots \\
\varphi_N^F \\
\varphi_{-N}^F \\
\varphi_{-(N-1)}^F \\
\vdots \\
\varphi_{-1}^F \\
\end{bmatrix}
= \frac{1}{\sqrt{M}}
\begin{bmatrix}
(W^*)^{0 \cdot 0} & \cdots & (W^*)^{0 \cdot (M-1)} \\
\vdots & \ddots & \vdots \\
(W^*)^{(M-1) \cdot 0} & \cdots & (W^*)^{(M-1) \cdot (M-1)}
\end{bmatrix}
\begin{bmatrix}
\varphi_0 \\
\varphi_1 \\
\vdots \\
\varphi_N \\
\varphi_{-N} \\
\varphi_{-(N-1)} \\
\vdots \\
\varphi_{-1}
\end{bmatrix}
\]

Note that the negative index \(-k\) (for \(k = 1, \ldots, N\)) can be considered also as the positive index \(M - k\).
Representation of a Discrete Signal in the DFT Domain

The DFT-domain representation is obtained via

\[ \phi^F = [\text{DFT}] \phi \]

Multiplying both sides of the equation by \([\text{DFT}]^*\), i.e.,

\[ [\text{DFT}]^* \phi^F = [\text{DFT}]^* [\text{DFT}] \phi \]

and, as the DFT matrix is unitary, we get

\[ \phi = [\text{DFT}]^* \phi^F \]

which is the inverse DFT procedure:

Given \( \phi^F \) it provides the signal-domain representation \( \phi \).
Consider the following discrete signal of $M$ samples:

For $n = 0, \ldots, M - 1$:

$$\phi_n = \delta_{n,n_0} = \begin{cases} 1, & \text{for } n = n_0 \\ 0, & \text{otherwise} \end{cases},$$

where $n_0 \in \{0, \ldots, M - 1\}$.

$\delta_{n,n_0}$ is also known as the Kronecker delta, here shifted to $n_0$.

The DFT of the above signal is

$$\Phi^F = \frac{1}{\sqrt{M}} \begin{bmatrix} (W^*)^{0\cdot0} & \ldots & (W^*)^{0\cdot(M-1)} \\ \vdots & \ddots & \vdots \\ (W^*)^{(M-1)\cdot0} & \ldots & (W^*)^{(M-1)\cdot(M-1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix} (W^*)^{0\cdot n_0} \\ (W^*)^{1\cdot n_0} \\ \vdots \\ (W^*)^{(M-1)\cdot n_0} \end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix} e^{-i\frac{2\pi}{M}0\cdot n_0} \\ e^{-i\frac{2\pi}{M}1\cdot n_0} \\ \vdots \\ e^{-i\frac{2\pi}{M}(M-1)\cdot n_0} \end{bmatrix}$$

Note the particular case of $n_0 = 0$. 

The $(n_0 + 1)^{th}$ entry

The $(n_0 + 1)^{th}$ column of the DFT matrix
DFT Example #2: Cosine Signal

Consider the following discrete signal of $M$ samples:

For $n = 0, ..., M - 1$:  
\[\varphi_n = \cos\left(\frac{2\pi k_0}{M} n\right)\]

where \(k_0 \in \{0, ..., M - 1\}\)

Recall that  
\[\cos\left(\frac{2\pi k_0}{M} n\right) = \frac{1}{2} e^{i\frac{2\pi k_0}{M} n} + \frac{1}{2} e^{-i\frac{2\pi k_0}{M} n} = \frac{1}{2} (W^{k_0 n} + W^{-k_0 n})\]

The $k^{th}$ component of the DFT-domain representation of the above signal is

\[\varphi^F_k = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} (W^*)^{k \cdot n} \varphi_n = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} W^{-k \cdot n} \frac{1}{2} (W^{k_0 n} + W^{-k_0 n}) = \]

\[= \frac{1}{\sqrt{M}} \left( \frac{1}{2} \sum_{n=0}^{M-1} W^{-k \cdot n} W^{k_0 n} + \frac{1}{2} \sum_{n=0}^{M-1} W^{-k \cdot n} W^{-k_0 n} \right) = \frac{1}{\sqrt{M}} \left( \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k-k_0) \cdot n} + \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k+k_0) \cdot n} \right)\]
DFT Example #2: Cosine Signal

Let us examine the expression
\[
\sum_{n=0}^{M-1} W^{-(k-k_0)n}.
\]

For \(k = k_0\):
\[
\sum_{n=0}^{M-1} W^{-(k-k_0)n} = \sum_{n=0}^{M-1} W^{-0n} = \sum_{n=0}^{M-1} 1 = M
\]

For \(k \neq k_0\):
\[
\sum_{n=0}^{M-1} W^{-(k-k_0)n} = \sum_{n=0}^{M-1} (W^{-(k-k_0)})^n = \frac{(W^{-(k-k_0)})^M - 1}{W^{-(k-k_0)} - 1}
\]

Recall that \(W = e^{\frac{2\pi i}{M}}\) [and that \(W^0, W^1, \ldots, W^{M-1}\) are the \(M\) roots (of order \(M\)) of unity].

Noting that \((W^{-(k-k_0)})^M = (W^M)^{-(k-k_0)} = (e^{\frac{2\pi i}{M}M})^{-(k-k_0)} = (e^{2\pi})^{-(k-k_0)} = 1\) implies

For \(k \neq k_0\):
\[
\sum_{n=0}^{M-1} W^{-(k-k_0)n} = 0
\]

\[
\sum_{n=0}^{M-1} W^{-(k-k_0)n} = M \cdot \delta_{k,k_0} \triangleq \begin{cases} \frac{M}{k=k_0}, & \text{for } k = k_0 \\ 0, & \text{otherwise} \end{cases}
\]
DFT Example #2: Cosine Signal

Using the last result

\[
\sum_{n=0}^{M-1} W^{-(k-k_0)\cdot n} = M \cdot \delta_{k,k_0} \triangleq \begin{cases} M, & \text{for } k = k_0 \\ 0, & \text{otherwise} \end{cases}
\]

The following development justifies the correspondence between the negative index \(-k_0\) and the index \(M - k_0\):

\[
\sum_{n=0}^{M-1} W^{-(k+k_0)\cdot n} = \sum_{n=0}^{M-1} W^{-(k+k_0-M)\cdot n} = \sum_{n=0}^{M-1} W^{-(k-(M-k_0))\cdot n} = M \cdot \delta_{k,M-k_0} \triangleq \begin{cases} M, & \text{for } k = M - k_0 \\ 0, & \text{otherwise} \end{cases}
\]

We develop the expression for the \(k^{th}\) component of the DFT-domain representation of the cosine signal:

\[
\phi_k^F = \frac{1}{\sqrt{M}} \left( \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k-k_0)\cdot n} + \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k+k_0)\cdot n} \right) = \frac{1}{\sqrt{M}} \left( \frac{1}{2} M \cdot \delta_{k,k_0} + \frac{1}{2} M \cdot \delta_{k,M-k_0} \right) =
\]

\[
= \frac{\sqrt{M}}{2} \delta_{k,k_0} + \frac{\sqrt{M}}{2} \delta_{k,M-k_0}
\]
Image Enhancement in The DFT Domain

• We are given a noisy image of size $256 \times 256$:

$$I_{\text{noisy}}[r, n] = I[r, n] + \text{noise}[r, n]$$

• The noise is **harmonic** and follows the formula:

$$\text{noise}[r, n] = A_r \cdot \cos \left( 2\pi fn + \phi_r \right)$$

• $f = \frac{1}{8 \text{ pixels}}$

• The amplitude, $A$, and the phase, $\varphi$, are **random** and independent for each line.
Image Enhancement in The DFT Domain

\[ A_{100} = 22.37 \]

\[ \varphi_{100} = 1.325\text{rad} \]
Image Enhancement in The DFT Domain

Original Cameraman Image

Picture with noise
Image Enhancement in The DFT Domain
The Image-Domain Smoothing Alternative

Picture after average filter 1X4

Picture after average filter 1X6
Image Enhancement in The DFT Domain
Alternatives: Smoothing vs Median (8 pixels)

Picture after average filter 1x8

No noise but image is blurred
Image Enhancement in The DFT Domain

- DFT of the noise in line \( r \)

Recall that \( M = 256 \) and \( f = \frac{1}{8} \), hence

\[
noise_n^{(r)} = A_r \cos(2\pi fn + \phi_r) = A_r \cos \left(2\pi \frac{32}{N} n + \phi_r \right)
\]

Then, since the signal is a shifted cosine function, its DFT is

\[
DFT \left\{ noise_n^{(r)} \right\}_k = \begin{cases} 
\frac{\sqrt{M}}{2} A_r e^{i\phi_r}, & k = 32, 224 \\
0, & \text{else}
\end{cases}
\]

Here we would like to handle frequencies 32 and 224
(recall that 224 can also be considered as -32).
Image Enhancement in The DFT Domain

Noisy signal in DFT domain

Filtered signal in DFT domain

Notch Filter: Attenuate Specific Frequencies
Image Enhancement in The DFT Domain

- The noise was significantly removed.
- Original image was not fully restored
  - We cannot restore the attenuated frequencies
Image Enhancement in The DFT Domain

Notch filter

Smoothing filter of 8 pixels
Image Enhancement in The DFT Domain Implementation

- Filter in freq. domain:
  \[ \text{Filter} = \text{ones}(1,256); \]
  \[ \text{Filter}(32+1)=0; \]
  \[ \text{Filter}(224+1)=0; \]

- Filtration:
  \[ \text{For } k=1:\text{size}(I,1), \]
  \[ \quad Y = \text{fft}(I(k,:)).*\text{Filter}; \]
  \[ \quad I(k,:) = \text{ifft}(Y); \]
  \[ \text{end} \]