Signals and Systems

Tutorial 10

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Input signal: $\varphi^{in}(t)$ defined for $t \in (-\infty, \infty)$.

$\varphi^{in}(t)$ can be a periodic extension of a signal defined over a limited interval, e.g.,

the periodic extension of $\varphi^{in}_{L}(t)$, $t \in [0,1)$, is defined for any integer $T$ and $t \in [0,1)$ as $\varphi^{in}(t + T) = \varphi^{in}_{L}(t)$.

Output signal: $\varphi^{out}(t) \triangleq \mathcal{H}\{\varphi^{in}\}_{(t)}$ is defined for $t \in (-\infty, \infty)$. 
Linear Systems

**Linearity:**

\[
\mathcal{H}\{a_1 \cdot \varphi_1(t) + a_2 \cdot \varphi_2(t)\} = a_1 \cdot \mathcal{H}\{\varphi_1(t)\} + a_2 \cdot \mathcal{H}\{\varphi_2(t)\}
\]

**Is** \(\varphi^{\text{out}}(t) = \varphi^{\text{in}}(t - 1)\) **a linear system?**

\[
\mathcal{H}\{a_1 \cdot \varphi_1(t) + a_2 \cdot \varphi_2(t)\} = a_1 \cdot \varphi_1(t - 1) + a_2 \cdot \varphi_2(t - 1)
= a_1 \cdot \mathcal{H}\{\varphi_1(t)\} + a_2 \cdot \mathcal{H}\{\varphi_2(t)\}
\]

Hence, \(\varphi^{\text{out}}(t) = \varphi^{\text{in}}(t - 1)\) is linear.
Is $\varphi^{\text{out}}(t) = \varphi^{\text{in}}(t) + 1$ a linear system?

$\mathcal{H}\{a_1 \cdot \varphi_1(t) + a_2 \cdot \varphi_2(t)\} = a_1 \cdot \varphi_1(t) + a_2 \cdot \varphi_2(t) + 1$

and

$a_1 \cdot \mathcal{H}\{\varphi_1(t)\} + a_2 \cdot \mathcal{H}\{\varphi_2(t)\} = a_1 \cdot (\varphi_1(t) + 1) + a_2 \cdot (\varphi_2(t) + 1)$

$\rightarrow \mathcal{H}\{a_1 \cdot \varphi_1(t) + a_2 \cdot \varphi_2(t)\} \neq a_1 \cdot \mathcal{H}\{\varphi_1(t)\} + a_2 \cdot \mathcal{H}\{\varphi_2(t)\}$

Therefore, $\varphi^{\text{out}}(t) = \varphi^{\text{in}}(t) + 1$ is not linear.
Shift-Invariant Systems

The shifting operator: \( T_{t_0} \{ f(t) \} \triangleq f(t - t_0) \)

**Shift invariance:** \( \mathcal{H} \{ T_{t_0} \{ \varphi^{in}(t) \} \} = T_{t_0} \{ \mathcal{H} \{ \varphi^{in}(t) \} \} \)

For an input signal that is shifted by \( t_0: \varphi^{in}(t - t_0) \), the system’s output-signal will be also shifted in the same amount: \( \varphi^{out}(t - t_0) \).

Is \( \varphi^{out}(t) = \left( \varphi^{in}(t) \right)^2 \) a shift-invariant system?

\[
T_{t_0} \{ \mathcal{H} \{ \varphi^{in}(t) \} \} = \varphi^{out}(t - t_0) = \left( \varphi^{in}(t - t_0) \right)^2 = \left( T_{t_0} \{ \varphi^{in}(t) \} \right)^2 = \mathcal{H} \left\{ T_{t_0} \{ \varphi^{in}(t) \} \right\}
\]

Hence, \( \varphi^{out}(t) = \left( \varphi^{in}(t) \right)^2 \) has the shift-invariance property.
Is \( \varphi^{out}(t) = \varphi^{in}(t) + t \) a shift-invariant system?

Let us define \( \varphi_{shift}(t) = T_{t_0}\{\varphi^{in}(t)\} \) then

\[
\mathcal{H}\left\{T_{t_0}\{\varphi^{in}(t)\}\right\} = \mathcal{H}\{\varphi_{shift}(t)\} = \varphi_{shift}(t) + t = \varphi^{in}(t - t_0) + t
\]

and

\[
T_{t_0}\left\{\mathcal{H}\{\varphi^{in}(t)\}\right\} = T_{t_0}\{\varphi^{out}(t)\} = \varphi^{out}(t - t_0) = \varphi^{in}(t - t_0) + t - t_0
\]

hence

\[
\mathcal{H}\left\{T_{t_0}\{\varphi^{in}(t)\}\right\} \neq T_{t_0}\left\{\mathcal{H}\{\varphi^{in}(t)\}\right\}
\]

Therefore, \( \varphi^{out}(t) = \varphi^{in}(t) + t \) is not a shift-invariant system.
Is \( \varphi^{out}(t) = \begin{cases} \varphi^{in}(t), & 0 \leq t < 1 \\ 0, & \text{else} \end{cases} \) a linear and/or shift-invariant system?

**Linearity:**

\[ \mathcal{H}\{a_1 \cdot \varphi_1(t) + a_2 \cdot \varphi_2(t)\} = \begin{cases} a_1 \cdot \varphi_1(t) + a_2 \cdot \varphi_2(t), & 0 \leq t < 1 \\ 0, & \text{else} \end{cases} \]

and

\[ \rightarrow \mathcal{H}\{a_1 \cdot \varphi_1(t) + a_2 \cdot \varphi_2(t)\} = a_1 \cdot \mathcal{H}\{\varphi_1(t)\} + a_2 \cdot \mathcal{H}\{\varphi_2(t)\} \]

Therefore, the system is linear.
Is \( \varphi^{out}(t) = \begin{cases} \varphi^{in}(t) & , 0 \leq t < 1 \\ 0 & , \text{else} \end{cases} \) a linear and/or shift-invariant system?

**Shift invariance:**

Let us define \( \varphi^{in}_{shift}(t) = T_{t_0}\{\varphi^{in}(t)\} \)
then

\[
\mathcal{H}\{T_{t_0}\{\varphi^{in}(t)\}\} = \mathcal{H}\{\varphi^{in}_{shift}(t)\} = \begin{cases} \varphi^{in}_{shift}(t) & , 0 \leq t < 1 \\ 0 & , \text{else} \end{cases}
= \begin{cases} \varphi^{in}(t-t_0) & , 0 \leq t < 1 \\ 0 & , \text{else} \end{cases}
\]

whereas

\[
T_{t_0}\{\mathcal{H}\{\varphi^{in}(t)\}\} = T_{t_0}\{\varphi^{out}(t)\} = \varphi^{out}(t-t_0) = \begin{cases} \varphi^{in}(t-t_0) & , 0 \leq t-t_0 < 1 \\ 0 & , \text{else} \end{cases}
= \begin{cases} \varphi^{in}(t-t_0) & , t_0 \leq t < t_0 + 1 \\ 0 & , \text{else} \end{cases}
\]
Is \( \varphi^{out}(t) = \begin{cases} \varphi^{in}(t) & , 0 \leq t < 1 \\ 0 & , else \end{cases} \) a linear and/or shift-invariant system?

**Shift invariance:**

We got that

\[ \mathcal{H} \left\{ T_{t_0} \{ \varphi^{in}(t) \} \right\} \neq T_{t_0} \left\{ \mathcal{H} \{ \varphi^{in}(t) \} \right\} \]

Therefore, the system is not shift invariant.
The shift variance of $\varphi_{\text{out}}(t) = \begin{cases} 
\varphi_{\text{in}}(t), & 0 \leq t < 1 \\
0, & \text{else}
\end{cases}$ demonstrated for a linear input signal.

Regular settings:
The shift variance of $\varphi^{out}(t) = \begin{cases} \varphi^{in}(t) & , 0 \leq t < 1 \\ 0 & , else \end{cases}$ demonstrated for a linear input signal.

Shifted input signal:
System Properties: Example

The shift variance of $\varphi_{\text{out}}(t) = \begin{cases} \varphi_{\text{in}}(t) & , 0 \leq t < 1 \\ 0 & , \text{else} \end{cases}$ demonstrated for a linear input signal.

Shifted output signal:
Discrete-Time Systems

Consider discrete-time signals and systems defined for \( n \in \{0,1,\ldots,N-1\} \) for some integer \( N > 1 \).

Here \( x[n] \) is the input signal and \( y[n] \) is the output signal of the system. This signals also take the following vector form:

\[
\begin{bmatrix}
x[0] \\
x[1] \\
\vdots \\
x[N-1]
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
y[0] \\
y[1] \\
\vdots \\
y[N-1]
\end{bmatrix}
\]

If \( y = Hx \) holds for some \( N \times N \) matrix \( H \), the system is \textbf{linear}.

In case that \( H \) is also a circulant matrix, then the linear system is also \textbf{shift-invariant}.
Discrete-Time Systems

Determine whether the following system is linear and/or shift-invariant.

**System #1:**

\[ y[n] = \begin{cases} 
  x[n] & , n \text{ (mod 2)} = 0 \\
  0 & , else 
\end{cases} \]

Clearly this is a **linear system** (prove it by yourself).

It can be formulated in a matrix form as

\[
\begin{bmatrix}
  y[0] \\
  y[1] \\
  \vdots \\
  y[N - 1]
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
  x[1] \\
  \vdots \\
  x[N - 1]
\end{bmatrix}
\]

\( H \) is a diagonal matrix with alternating values of ones and zeros, hence, it is not circulant \( \rightarrow \) the system is **not shift invariant**.
Discrete-Time Systems

Determine whether the following system is linear and/or shift-invariant.

System #2:

\[ y[n] = x[n + 1 \mod N] - x[n] \]

This is a linear system (prove it by yourself).
It can be formulated in a matrix form as

\[
\begin{bmatrix}
  y[0] \\
y[1] \\
\vdots \\
y[N - 1]
\end{bmatrix} =
\begin{bmatrix}
  -1 & 1 & 1 \\
  -1 & 1 & 1 \\
  \vdots & \ddots & \ddots \\
  -1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
  x[0] \\
x[1] \\
\vdots \\
x[N - 1]
\end{bmatrix}
\]

\( H \) has two fixed-valued diagonals forming a circulant matrix structure → the system is **shift invariant**.
Discrete-Time Systems

Determine whether the following system is linear and/or shift-invariant.

System #3:

\[ y[n] = \sum_{k=0}^{n} x[k] \]

This is a linear system (prove it by yourself).
It can be formulated in a matrix form as

\[
\begin{bmatrix}
    y[0] \\
    y[1] \\
    \vdots \\
    y[N-1]
\end{bmatrix}
= 
\begin{bmatrix}
    1 & 0 & 0 & 0 & \cdots & 0 \\
    1 & 1 & 0 & 0 & \cdots & 0 \\
    1 & 1 & 1 & 0 & \cdots & 0 \\
    1 & 1 & 1 & 1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
    x[0] \\
    x[1] \\
    \vdots \\
    x[N-1]
\end{bmatrix}
\]

\(H\) is a lower triangular matrix of ones, hence, it is a not a circulant matrix → the system is not shift invariant.
Determine whether the following system is linear and/or shift-invariant.

System #4:

\[
y[n] = \sum_{k=0}^{M} x[n - k \ (\text{mod} \ N)]
\]

This is a linear system (prove it by yourself).

It can be formulated in a matrix form as (example for \(M = 3\))

\[
\begin{bmatrix}
    y[0] \\
    y[1] \\
    \vdots \\
    y[N-1]
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & \cdots & 0 & 1 & 1 \\
    1 & 1 & 0 & \cdots & 0 & 1 \\
    1 & 1 & 1 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
    x[0] \\
    x[1] \\
    \vdots \\
    x[N-1]
\end{bmatrix}
\]

\(H\) is a circulant matrix
→ the system is shift invariant.