Signal, Image and Data Processing (236200)

Tutorial 7

The Discrete Fourier Transform (DFT)
The DFT Matrix

The DFT matrix of size $M \times M$ is defined as

$$[\text{DFT}] = \frac{1}{\sqrt{M}} \begin{bmatrix} (W^*)^{0\cdot0} & \cdots & (W^*)^{0\cdot(M-1)} \\ \vdots & \ddots & \vdots \\ (W^*)^{(M-1)\cdot0} & \cdots & (W^*)^{(M-1)\cdot(M-1)} \end{bmatrix}$$

where $W \triangleq e^{i\frac{2\pi}{M}} = \cos\left(\frac{2\pi}{M}\right) + i \sin\left(\frac{2\pi}{M}\right)$

and $i \triangleq \sqrt{-1}$

and $^*$ denotes complex conjugate

$$W^* \triangleq e^{-i\frac{2\pi}{M}} = \cos\left(\frac{2\pi}{M}\right) - i \sin\left(\frac{2\pi}{M}\right)$$

The DFT matrix is symmetric and unitary, hence, its inverse is

$$[\text{DFT}]^* = \frac{1}{\sqrt{M}} \begin{bmatrix} W^{0\cdot0} & \cdots & W^{0\cdot(M-1)} \\ \vdots & \ddots & \vdots \\ W^{(M-1)\cdot0} & \cdots & W^{(M-1)\cdot(M-1)} \end{bmatrix}$$

i.e., $[\text{DFT}]^*[\text{DFT}] = [\text{DFT}][\text{DFT}]^* = I$
Representation of a Discrete Signal in the DFT Domain

Uniform sampling of the continuous signal \( \varphi(t) \) provides us the discrete set of \( M \equiv 2N + 1 \) samples:

\[
\varphi_{-N}, \varphi_{-(N-1)}, \ldots, \varphi_0, \ldots, \varphi_{N-1}, \varphi_N
\]

that can be also arranged in a column vector as follows:

\[
\varphi = \begin{bmatrix}
\varphi_0 \\
\varphi_1 \\
\vdots \\
\varphi_N \\
\varphi_{-N} \\
\varphi_{-(N-1)} \\
\vdots \\
\varphi_{-1}
\end{bmatrix}
\]

Note the position of the negative-indexed samples.
Representation of a Discrete Signal in the DFT Domain

The representation of the discrete signal \( \varphi \) is

\[
\varphi^F = [\text{DFT}] \varphi
\]

or in a more explicit form:

\[
\begin{bmatrix}
\varphi_0^F \\
\varphi_1^F \\
\vdots \\
\varphi_N^F \\
\varphi_{-(N-1)}^F \\
\vdots \\
\varphi_{-1}^F
\end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix}
(W^*)^{0\cdot0} & \cdots & (W^*)^{0\cdot(M-1)} \\
\vdots & \ddots & \vdots \\
(W^*)^{(M-1)\cdot0} & \cdots & (W^*)^{(M-1)\cdot(M-1)}
\end{bmatrix} \begin{bmatrix}
\varphi_0 \\
\varphi_1 \\
\vdots \\
\varphi_N \\
\varphi_{-(N-1)} \\
\vdots \\
\varphi_{-1}
\end{bmatrix}
\]

Note that the negative index \(-k\) (for \(k = 1, \ldots, N\)) can be considered also as the positive index \(M - k\).
Representation of a Discrete Signal in the DFT Domain

The DFT-domain representation is obtained via

$$\varphi^F = [\text{DFT}] \varphi$$

Multiplying both sides of the equation by $[\text{DFT}]^*$, i.e.,

$$[\text{DFT}]^* \varphi^F = [\text{DFT}]^* [\text{DFT}] \varphi$$

and, as the DFT matrix is unitary, we get

$$\varphi = [\text{DFT}]^* \varphi^F$$

which is the inverse DFT procedure:
Given $\varphi^F$ it provides the signal-domain representation $\varphi$. 
DFT Example #1: The Kronecker Delta

Consider the following discrete signal of \( M \) samples:

For \( n = 0, \ldots, M - 1 \):
\[
\varphi_n = \delta_{n,n_0} \triangleq \begin{cases} 
1, & \text{for } n = n_0 \\
0, & \text{otherwise}
\end{cases}
\]

where \( n_0 \in \{0, \ldots, M - 1\} \).

\( \delta_{n,n_0} \) is also known as the Kronecker delta, here shifted to \( n_0 \).

The DFT of the above signal is

\[
\varphi^F = \frac{1}{\sqrt{M}} \begin{bmatrix}
(W^*)^{0\cdot0} & \cdots & (W^*)^{0\cdot(M-1)} \\
\vdots & \ddots & \vdots \\
(W^*)^{(M-1)\cdot0} & \cdots & (W^*)^{(M-1)\cdot(M-1)}
\end{bmatrix}
\]

The \((n_0 + 1)^{th}\) entry

The \((n_0 + 1)^{th}\) column of the DFT matrix

\[
\frac{1}{\sqrt{M}} \begin{bmatrix}
(W^*)^{0\cdot0} \cdot n_0 \\
\vdots \\
(W^*)^{(M-1)\cdot0} \cdot n_0
\end{bmatrix}
= \frac{1}{\sqrt{M}} \begin{bmatrix}
e^{-i\frac{2\pi}{M}0\cdot n_0} \\
e^{-i\frac{2\pi}{M}1\cdot n_0} \\
\vdots \\
e^{-i\frac{2\pi}{M}(M-1)\cdot n_0}
\end{bmatrix}
\]

Note the particular case of \( n_0 = 0 \).
DFT Example #2: Cosine Signal

Consider the following discrete signal of $M$ samples:

For $n = 0, \ldots, M - 1$:

$$\varphi_n = \cos\left(\frac{2\pi k_0}{M} n\right)$$

where $k_0 \in \{0, \ldots, M - 1\}$

Recall that

$$\cos\left(\frac{2\pi k_0}{M} n\right) = \frac{1}{2} e^{i\frac{2\pi k_0}{M} n} + \frac{1}{2} e^{-i\frac{2\pi k_0}{M} n} = \frac{1}{2} (W^{k_0} n + W^{-k_0} n)$$

The $k^{th}$ component of the DFT-domain representation of the above signal is

$$\varphi_k^F = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} (W^*)^{k \cdot n} \varphi_n = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} W^{-k\cdot n} \frac{1}{2} (W^{k_0} n + W^{-k_0} n) =$$

$$= \frac{1}{\sqrt{M}} \left(\frac{1}{2} \sum_{n=0}^{M-1} W^{-k\cdot n} W^{k_0} n + \frac{1}{2} \sum_{n=0}^{M-1} W^{-k\cdot n} W^{-k_0} n\right) = \frac{1}{\sqrt{M}} \left(\frac{1}{2} \sum_{n=0}^{M-1} W^{-(k-k_0)\cdot n} + \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k+k_0)\cdot n}\right)$$
DFT Example #2: Cosine Signal

Let us examine the expression \( \sum_{n=0}^{M-1} W^{-(k-k_0) \cdot n} \):

For \( k = k_0 \):
\[
\sum_{n=0}^{M-1} W^{-(k-k_0) \cdot n} = \sum_{n=0}^{M-1} W^{-0 \cdot n} = \sum_{n=0}^{M-1} 1 = M
\]

For \( k \neq k_0 \):
\[
\sum_{n=0}^{M-1} W^{-(k-k_0) \cdot n} = \sum_{n=0}^{M-1} (W^{-(k-k_0)})^n = \frac{(W^{-(k-k_0)})^M - 1}{W^{-(k-k_0)} - 1}
\]

Recall that \( W = e^{i \frac{2\pi}{M}} \) [and that \( W^0, W^1, \ldots, W^{M-1} \) are the \( M \) roots (of order \( M \)) of unity].

Noting that \( (W^{-(k-k_0)})^M = (W^M)^{-(k-k_0)} = (e^{i \frac{2\pi}{M}})^{-(k-k_0)} = (e^{i 2\pi})^{-(k-k_0)} = 1 \) implies

For \( k \neq k_0 \):
\[
\sum_{n=0}^{M-1} W^{-(k-k_0) \cdot n} = 0
\]

\[
\sum_{n=0}^{M-1} W^{-(k-k_0) \cdot n} = M \cdot \delta_{k,k_0} = \begin{cases} M & \text{for } k = k_0 \\ 0 & \text{otherwise} \end{cases}
\]
DFT Example #2: Cosine Signal

Using the last result

\[
\sum_{n=0}^{M-1} W^{-(k-k_0) \cdot n} = M \cdot \delta_{k,k_0} \triangleq \begin{cases} M & \text{for } k = k_0 \\ 0 & \text{otherwise} \end{cases}
\]

The following development justifies the correspondence between the negative index \(-k_0\) and the index \(M - k_0\):

\[
\sum_{n=0}^{M-1} W^{-(k+k_0) \cdot n} = \sum_{n=0}^{M-1} W^{-(k+k_0-M) \cdot n} = \sum_{n=0}^{M-1} W^{-(k-(M-k_0)) \cdot n} = M \cdot \delta_{k,M-k_0} \triangleq \begin{cases} M & \text{for } k = M - k_0 \\ 0 & \text{otherwise} \end{cases}
\]

We develop the expression for the \(k^{th}\) component of the DFT-domain representation of the cosine signal:

\[
\varphi_k^k = \frac{1}{\sqrt{M}} \left( \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k-k_0) \cdot n} + \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k+k_0) \cdot n} \right) = \frac{1}{\sqrt{M}} \left( \frac{1}{2} M \cdot \delta_{k,k_0} + \frac{1}{2} M \cdot \delta_{k,M-k_0} \right) = \\
\frac{\sqrt{M}}{2} \delta_{k,k_0} + \frac{\sqrt{M}}{2} \delta_{k,M-k_0}
\]
Image Enhancement in The DFT Domain

• We are given a noisy image of size $256 \times 256$:

$$I_{noisy}[r,n] = I[r,n] + noise[r,n]$$

• The noise is harmonic and follows the formula:

$$noise[r,n] = A_r \cdot \cos(2\pi fn + \phi_r)$$

• $f = \frac{1}{8 \text{ pixels}}$

• The amplitude, $A$, and the phase, $\varphi$, are random and independent for each row.
Image Enhancement in The DFT Domain

\[ A_{100} = 22.37 \]
\[ \varphi_{100} = 1.325 \text{ rad} \]
Image Enhancement in The DFT Domain
Image Enhancement in The DFT Domain
The Image-Domain Smoothing Alternative
Image Enhancement in The DFT Domain

Alternatives: Smoothing vs Median (8 pixels)

Picture after average filter 1x8

Picture after median filter 1x8

No noise but image is blurred

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Image Enhancement in The DFT Domain

• DFT of the noise in line $r$

Recall that $M = 256$ and $f = \frac{1}{8}$, hence

$$noise_n^{(r)} = A_r \cos (2\pi fn + \phi_r) = A_r \cos \left( 2\pi \frac{32}{M} n + \phi_r \right)$$

Then, since the signal is a shifted cosine function, its DFT is

$$DFT \left\{ noise^{(r)} \right\}_k = \begin{cases} \sqrt{M} \over 2 & A_r e^{i\phi_r}, \ k = 32, 224 \\ 0 & \text{else} \end{cases}$$

Here we would like to handle frequencies 32 and 224 (recall that 224 can also be considered as -32).
Image Enhancement in The DFT Domain

Noisy signal in DFT domain

Filtered signal in DFT domain

Notch Filter: Attenuate Specific Frequencies
Image Enhancement in The DFT Domain

- The noise was significantly removed.
- Original image was not fully restored
  - We cannot restore the attenuated frequencies
Image Enhancement in The DFT Domain

Notch filter

Smoothing filter of 8 pixels
• Filter in freq. domain:
  Filter=ones(1,256);
  Filter(32+1)=0;
  Filter(224+1)=0;

• Filtration:
  For k=1:size(I,1),
    Y=fft(I(k,:)).*Filter;
    I(k,:)=ifft(Y);
end
DFT Example #3: Periodic Delta Signal

Consider the following discrete signal of $N$ samples:

For $n = 0, ..., N - 1$:

$$\varphi_n = \begin{cases} 
1 & \text{for } n = 0, T, ..., (c - 1)T \\
0 & \text{otherwise}
\end{cases}$$

where $N = cT$ for some positive integer $c$.

What is the DFT of $\varphi$?

**Solution:**

Using the definition of Kronecker’s delta

$$\delta_{n,n_0} = \begin{cases} 
1 & , \text{for } n = n_0 \\
0 & , \text{otherwise}
\end{cases}$$

we can write the signal as

$$\varphi_n = \sum_{l=0}^{c-1} \delta_{n,Tl}$$

where $T$ and $c$ were defined in the question.
DFT Example #3: Periodic Delta Signal

Recall the definition of the $N^{th}$ order root of the unity: $W_N = e^{\frac{i2\pi}{N}}$.

Then the $k^{th}$ DFT coefficient is

$$\phi_k^F = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (W_N^*)^{k \cdot n} \varphi_n = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left( (W_N^*)^{k \cdot n} \sum_{l=0}^{c-1} \delta_{n,Tl} \right) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \sum_{l=0}^{c-1} \delta_{n,Tl} \cdot (W_N^*)^{k \cdot n}$$

$$= \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_N^*)^{k \cdot Tl}$$

Since $N = cT$ we get that $(W_N^*)^{k \cdot Tl} = \left( e^{-\frac{i2\pi}{N}} \right)^{k \cdot Tl} = e^{-\frac{i2\pi}{N} \cdot k \cdot Tl} = e^{-\frac{i2\pi}{c} \cdot kl} = (W_c^*)^{kl}$

and therefore

$$\phi_k^F = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_N^*)^{k \cdot Tl} = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_c^*)^{kl}$$
DFT Example #3: Periodic Delta Signal

For $k = 0, c, ..., (T - 1)c$ we get that

$$\frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_c^*)^{kl} = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_c^*)^k = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (1)^l = \frac{1}{\sqrt{N}} c = \frac{1}{\sqrt{N}} \cdot \frac{N}{T} = \frac{\sqrt{N}}{T}$$

For $k \neq 0, c, ..., (T - 1)c$ we calculate the sum of the geometric series as

$$\sum_{l=0}^{c-1} (W_c^*)^{kl} = \frac{1 - (W_c^*)^k}{1 - (W_c^*)^k} = \frac{1 - (W_c^*)^c}{1 - (W_c^*)^k} = \frac{1 - 1}{1 - (W_c^*)^k} = 0$$

where we used the fact that $(W_c^*)^c = \left(e^{-i2\pi/c}\right)^c = e^{-i2\pi} = 1$.

To conclude, we got that

$$\varphi_k^F = \begin{cases} \frac{\sqrt{N}}{T} & \text{for } k = 0, c, ..., (T - 1)c \\ 0 & \text{otherwise} \end{cases}$$