Signal, Image, and Data Processing (236200)

Tutorial 11

Pseudo-Inverse Filtering for Linear Shift-Invariant Degradations
Restoration from *Noiseless* Degradation

• In many practical systems a disturbing component is deteriorates the “clean” signal

• This component is often modeled as a degradation operator.
• Examples for common image degradations are blur, resolution reduction, and pixel erasure.

\[
\begin{align*}
\varphi \in \mathbb{R}^N & \quad & H \in \mathbb{R}^{M \times N} & \quad & \varphi^{DATA} \triangleq H\varphi \in \mathbb{R}^M
\end{align*}
\]

The task is to estimate the unknown signal \( \varphi \) from the given degraded measurement \( \varphi^{DATA} \). We assume that \( H \) is known.
In this tutorial $\bar{\phi}$ is deterministic (i.e., no probabilistic modeling is used here).

**Restoration from *Noiseless* Degradation**

The task is to estimate the unknown signal $\bar{\phi}$ from the given degraded measurement $\bar{\phi}^{DATA}$. We assume that $\mathbf{H}$ is known.

\[
\bar{\phi} \in \mathbb{R}^N \quad \quad \mathbf{H} \in \mathbb{R}^{M \times N} \quad \quad \bar{\phi}^{DATA} = \mathbf{H}\bar{\phi} \in \mathbb{R}^M
\]

\[
\hat{\phi} = \mathbf{M}\bar{\phi}^{DATA} \in \mathbb{R}^N \quad \quad \mathbf{M} \in \mathbb{R}^{N \times M}
\]
Inverse Filtering:
The Trivial Case of Invertible Degradation Operators

Here $H$ is invertible, thus, $M = H^{-1}$ leads to $\hat{\varphi} = H^{-1} \varphi^{DATA} = H^{-1}H\hat{\varphi} = \hat{\varphi}$ → Accurate recovery of the clean image.
Inverse Filtering: The Case of Non-Invertible Degradation Operators

What should be the optimal linear filter $\mathbf{M}$ when $\mathbf{H}$ is not invertible? (this is the common case in practice)
Inverse Filtering for Linear Shift-Invariant Degradation Operators

In these settings, $\mathbf{H}$ is a $N \times N$ circulant matrix.

Hence, it is diagonalized by the $[DFT]^*$ matrix, namely,

$$\mathbf{H} = [DFT]^* \Lambda_{\mathbf{H}} [DFT]$$

where $\Lambda_{\mathbf{H}}$ is a diagonal matrix with diagonal values $\{\lambda_k^H\}_{k=1}^N$.

Recall also that

$$\begin{bmatrix}
\lambda_1^H \\
\lambda_2^H \\
\lambda_3^H \\
\vdots \\
\lambda_N^H
\end{bmatrix} = [DFT]^* \begin{bmatrix}
h_0 \\
h_{N-1} \\
h_{N-2} \\
\vdots \\
h_1
\end{bmatrix}$$

The first row of $\mathbf{H}$ reordered as a column vector.
Inverse Filtering for Linear Shift-Invariant Degradation Operators

\[ \bar{\varphi}^{DATA} = H\bar{\varphi} = [DFT]^* \Lambda_H [DFT] \bar{\varphi} \]

\[ \rightarrow [DFT] \bar{\varphi}^{DATA} = \Lambda_H [DFT] \bar{\varphi} \]

\[ \begin{bmatrix} \varphi_{1,DATA,F}^F \\ \varphi_{2,DATA,F}^F \\ \vdots \\ \varphi_{N,DATA,F}^F \end{bmatrix} = \Lambda_H \begin{bmatrix} \varphi_1^F \\ \varphi_2^F \\ \vdots \\ \varphi_N^F \end{bmatrix} = \begin{bmatrix} \lambda_1^H \varphi_1^F \\ \lambda_2^H \varphi_2^F \\ \vdots \\ \lambda_N^H \varphi_N^F \end{bmatrix} \]

DFT domain components:

\[ \varphi_{k,DATA,F}^F = \lambda_k^H \varphi_k^F \]

\[ k = 1, \ldots, N \]

Using the definitions

\[ \begin{bmatrix} \varphi_1^F \\ \varphi_2^F \\ \vdots \\ \varphi_N^F \end{bmatrix} \triangleq [DFT] \bar{\varphi} \]

and

\[ \begin{bmatrix} \varphi_{1,DATA,F}^F \\ \varphi_{2,DATA,F}^F \\ \vdots \\ \varphi_{N,DATA,F}^F \end{bmatrix} \triangleq [DFT] \bar{\varphi}^{DATA} \]
Inverse Filtering for **Linear Shift-Invariant** Degradation Operators

Let us consider $\mathbf{M}$ which is **circulant**, having eigenvalues $\{\lambda_k^\mathbf{M}\}_{k=1}^N$:

$$\hat{\varphi} = \mathbf{M}\varphi^{DATA} = [DFT]^* \Lambda_{\mathbf{M}} [DFT] \varphi^{DATA}$$

$$\rightarrow [DFT]\hat{\varphi} = \Lambda_{\mathbf{M}} [DFT] \varphi^{DATA}$$

$$\rightarrow \begin{bmatrix} \hat{\phi}_1^F \\ \hat{\phi}_2^F \\ \vdots \\ \hat{\phi}_N^F \end{bmatrix} = \Lambda_{\mathbf{M}} \begin{bmatrix} \varphi_1^{DATA,F} \\ \varphi_2^{DATA,F} \\ \vdots \\ \varphi_N^{DATA,F} \end{bmatrix} = \begin{bmatrix} \lambda_1^\mathbf{M} \varphi_1^{DATA,F} \\ \lambda_2^\mathbf{M} \varphi_2^{DATA,F} \\ \vdots \\ \lambda_N^\mathbf{M} \varphi_N^{DATA,F} \end{bmatrix} \rightarrow \hat{\phi}^F_k = \lambda_k^\mathbf{M} \varphi_k^{DATA,F}$$

$k = 1, ..., N$

Recall that $\varphi_k^{DATA,F} = \lambda_k^\mathbf{H} \varphi_k^F$ and, therefore, $\hat{\phi}_k^F = \lambda_k^\mathbf{M} \lambda_k^\mathbf{H} \varphi_k^F$
Inverse Filtering for *Linear Shift-Invariant* Degradation Operators

We got that the restored signal having the DFT coefficients

$$\hat{\varphi}_k^F = \lambda_k^M \lambda_k^H \varphi_k^F, \; k = 1, \ldots, N$$

How should we design the circulant filter matrix $\mathbf{M}$?

Clearly, for $k$ where $\lambda_k^H \neq 0$ then we can set $\lambda_k^M = \frac{1}{\lambda_k^H}$ and get

$$\hat{\varphi}_k^F = \lambda_k^M \lambda_k^H \varphi_k^F = \frac{1}{\lambda_k^H} \lambda_k^H \varphi_k^F = \varphi_k^F$$

which is a **perfect recovery of the respective DFT coefficients** of the unknown clean signal.
Inverse Filtering for *Linear Shift-Invariant* Degradation Operators

The restored signal having the DFT coefficients:  
\[
\hat{\phi}_k^F = \lambda_k^M \lambda_k^H \varphi_k^F, \quad k = 1, \ldots, N
\]

For \( k \) where \( \lambda_k^H = 0 \) then we can observe that

\[
\hat{\phi}_k^F = \lambda_k^M \lambda_k^H \varphi_k^F = \lambda_k^M \cdot 0 \cdot \varphi_k^F = 0
\]

namely, the corresponding DFT coefficients in the restored signal are zeros independently of the choice of the respective filter coefficients \( \lambda_k^M \).

\[ \rightarrow \] we can just set \( \lambda_k^M = 0 \) for \( k \) where \( \lambda_k^H = 0 \).

The above defined filter is the *pseudo-inverse filter* in the case considered here.
Inverse Filtering for Linear Shift-Invariant Degradation Operators

In summary, the pseudo-inverse filter $\mathbf{M}$ is designed to be circulant with the eigenvalues:

$$\lambda^\mathbf{M}_k = \begin{cases} \frac{1}{\lambda^\mathbf{H}_k} & \text{for } k \text{ where } \lambda^\mathbf{H}_k \neq 0 \\ 0 & \text{for } k \text{ where } \lambda^\mathbf{H}_k = 0 \end{cases}$$

leading to an estimate with the following DFT coefficients:

$$\hat{\varphi}_k^F = \begin{cases} \varphi_k^F & \text{for } k \text{ where } \lambda^\mathbf{H}_k \neq 0 \\ 0 & \text{for } k \text{ where } \lambda^\mathbf{H}_k = 0 \end{cases}$$

Returning to the signal domain (in a vector form):

$$\hat{\varphi} = [DFT]^* \hat{\varphi}^F = \sum_{k=0}^{N} \hat{\varphi}_k^F \bar{\beta}_k$$

where $\bar{\beta}_k$ is the $k^{th}$ column of the $[DFT]^*$ matrix.
Inverse Filtering for *Linear Shift-Invariant* Degradation Operators

Using the result that \( \hat{\phi}_k = 0 \) for \( k \) where \( \lambda_k^H = 0 \) we get

\[
\hat{\varphi} = \sum_{k=0}^{N} \hat{\phi}_k F \beta_k = \sum_{k: \lambda_k^H \neq 0} \hat{\phi}_k F \beta_k = \sum_{k: \lambda_k^H \neq 0} \varphi_k F \beta_k
\]

The squared error of the estimation is

\[
\mathcal{E}^2 = \| \varphi - \hat{\varphi} \|_2^2 = \left\| \sum_{k=0}^{N} \varphi_k^F \beta_k - \sum_{k: \lambda_k^H \neq 0} \varphi_k^F \beta_k \right\|_2^2 = \left\| \sum_{k: \lambda_k^H = 0} \varphi_k^F \beta_k \right\|_2^2
\]

\[
= \left( \sum_{k: \lambda_k^H = 0} \varphi_k^F \beta_k \right)^* \left( \sum_{k: \lambda_k^H = 0} \varphi_k^F \beta_k \right) = \sum_{k: \lambda_k^H = 0} |\varphi_k|^2
\]

Note that the nullspace of the degradation operator \( H \) is spanned by the set of vectors \( \beta_k \) corresponding to \( k \) where \( \lambda_k^H = 0 \).
Inverse Filtering for *Linear Shift-Invariant* Degradation Operators

The above pseudo-inverse filtering procedure corresponds to the following constrained optimization problem:

\[
\hat{\varphi}_{\text{opt}} = \arg\min_{\hat{\varphi}} \|\hat{\varphi}\|_2^2 \\
\text{subject to } \mathbf{H}\hat{\varphi} = \varphi^{DATA}
\]

The above optimization looks for the shortest vector \( \hat{\varphi} \) that explains the given data \( \varphi^{DATA} \).

Let us show that

\[
\hat{\varphi} = \sum_{k: \lambda_k^H \neq 0} \varphi_k F \beta_k
\]

is the solution for the above optimization problem.
Inverse Filtering for *Linear Shift-Invariant* Degradation Operators

First, let’s examine the constraint: \( \mathbf{H} \hat{\phi} = \bar{\varphi}^{DATA} \)

\[
[DFT]^* \Lambda_{\mathbf{H}} [DFT][DFT]^* \begin{bmatrix}
\hat{\phi}_1^F \\
\hat{\phi}_2^F \\
\vdots \\
\hat{\phi}_N^F
\end{bmatrix} = [DFT]^* \begin{bmatrix}
\varphi_1^{DATA,F} \\
\varphi_2^{DATA,F} \\
\vdots \\
\varphi_N^{DATA,F}
\end{bmatrix}
\]

\(\Lambda_{\mathbf{H}}\begin{bmatrix}
\hat{\phi}_1^F \\
\hat{\phi}_2^F \\
\vdots \\
\hat{\phi}_N^F
\end{bmatrix} = \begin{bmatrix}
\varphi_1^{DATA,F} \\
\varphi_2^{DATA,F} \\
\vdots \\
\varphi_N^{DATA,F}
\end{bmatrix}
\]

\[
\lambda_k^H \hat{\phi}_k^F = \lambda_k^H \varphi_k^F
\]

\[
\begin{cases}
\hat{\phi}_k^F = \varphi_k^F & \text{for } k \text{ where } \lambda_k^H \neq 0 \\
\text{unconstrained} & \text{for } k \text{ where } \lambda_k^H = 0
\end{cases}
\]
Inverse Filtering for *Linear Shift-Invariant* Degradation Operators

Since the **pseudo-inverse estimate**

\[ \hat{\phi} = \sum_{k: \lambda_k^H \neq 0} \phi_k^F \beta_k \]

obeys

\[ \hat{\phi}_k^F = \begin{cases} \phi_k^F & \text{for } k \text{ where } \lambda_k^H \neq 0 \\ 0 & \text{for } k \text{ where } \lambda_k^H = 0 \end{cases} \]

it *satisfies* the optimization constraint:

\[ \begin{cases} \hat{\phi}_k^F = \phi_k^F & \text{for } k \text{ where } \lambda_k^H \neq 0 \\ \text{unconstrained} & \text{for } k \text{ where } \lambda_k^H = 0 \end{cases} \]
Inverse Filtering for *Linear Shift-Invariant* Degradation Operators

Clearly, any estimate of the form

\[
\hat{\Phi}_{\text{candidate}} = \sum_{k: \lambda_k^H \neq 0} \phi_k^F \bar{\beta}_k + \sum_{k: \lambda_k^H = 0} \eta_k^F \bar{\beta}_k
\]

that corresponds to

\[
\hat{\Phi}_k^F = \begin{cases} 
\phi_k^F & \text{for } k \text{ where } \lambda_k^H \neq 0 \\
\eta_k^F & \text{for } k \text{ where } \lambda_k^H = 0 
\end{cases}
\]

satisfies the optimization constraint:

\[
\begin{cases}
\hat{\Phi}_k^F = \phi_k^F & \text{for } k \text{ where } \lambda_k^H \neq 0 \\
\text{unconstrained} & \text{for } k \text{ where } \lambda_k^H = 0
\end{cases}
\]

Let’s show that the pseudo-inverse solution is the minimizer of \( \|\hat{\Phi}\|_2^2 \).
Inverse Filtering for *Linear Shift-Invariant* Degradation Operators

The signals obeying the constraint are of the form

\[
\hat{\Phi}_{\text{candidate}} = \sum_{k: \lambda^H_k \neq 0} \phi_k^F \bar{\beta}_k + \sum_{k: \lambda^H_k = 0} \eta_k^F \bar{\beta}_k
\]

correspond to the Euclidean vector-length of

\[
\|\hat{\Phi}\|_2^2 = \left\| \sum_{k: \lambda^H_k \neq 0} \phi_k^F \bar{\beta}_k + \sum_{k: \lambda^H_k = 0} \eta_k^F \bar{\beta}_k \right\|_2^2 = \sum_{k: \lambda^H_k \neq 0} |\phi_k^F|^2 + \sum_{k: \lambda^H_k = 0} |\eta_k^F|^2
\]

that is obviously minimized when \( \eta_k^F = 0 \) for \( k \) where \( \lambda^H_k = 0 \).

→ this property is achieved by the pseudo-inverse filter and, therefore, it is the minimizer of the optimization problem.