Statistical Data Processing (236200)

Winter 2017-18

Homework #4

Publication: January 15th, 2018
Submission: by January 25th at 23:59

Guidelines:

- Submission is in singles.
- Submit your solution to the TA mailbox at Taub floor 5.
- Matlab code should be electronically submitted via the course website.
- You may answer in Hebrew or English, in a clear-handwrite or a printed form.
- Follow the FAQ at course website, as clarifications and corrections may be published only there.

הנחיות:


-mitekha lanish b'hitdim.

הנגישים ידית לוח המשטחים האחראים בטאוב קומה 5.

אנ קוד המשטבים יש להגיש אלקטוריית בוחר הקורס.

mitekha lutanot bebeirah va b'eenot, bechber borah va meso.

נהמב לעוקב אחורי במאטר. הבاختות ותיקונים בקורס יפרסמו שיש בלבד.

Follow the FAQ at course website, as clarifications and corrections may be published only there.
**Question #1 (40 Points)**

A class of one-dimensional discrete signals is defined as follows.

A signal is the column vector of \( N \) samples, where \( N \) is an even number, of the form:

\[
\vec{\varphi} = [M, \ldots, M, M + L, M, \ldots, M, M + L, M, \ldots, M]^T
\]

i.e., all the vector components have the value \( M \) except for the components in the \( K \) and the \((K + \frac{N}{2})\) coordinates that have the value \( M + L \).

**Note that** \( K \in \{1, \ldots, \frac{N}{2}\} \).

The vector components are indexed starting at 1, i.e., the vector can be generally formulated as

\[
\vec{\varphi} = [\varphi_1, \ldots, \varphi_N]^T
\]

\( M, L, \) and \( K \) are independent random variables:

- \( K \) is a uniform random variable over the integers \( \{1, \ldots, \frac{N}{2}\} \).
- \( M \) obeys \( \mathbb{E}\{M\} = 0 \) and \( \mathbb{E}\{M^2\} = c \)
- \( L \) obeys \( \mathbb{E}\{L\} = 0 \) and \( \mathbb{E}\{L^2\} = \frac{N}{2}(1 - c) \)

where \( 0 < c < 1 \) is a (deterministic) constant.

a. Calculate the autocorrelation matrix of \( \vec{\varphi} \), denoted as \( \mathbf{R}_{\varphi} \), and show it is circulant.
   Note that the next subsections do not depend on your answer here.

b. Explain how the eigenvalues of \( \mathbf{R}_{\varphi} \) can be computed in a way that is simpler than an explicit diagonalization/eigendecomposition procedure.

c. Consider two general and independent random vectors \( \vec{\varphi}^{(1)} \) and \( \vec{\varphi}^{(2)} \), having the same size and zero-mean.
   The autocorrelation matrix of \( \vec{\varphi}^{(1)} \) is denoted as \( \mathbf{R}^{(1)} \), and the autocorrelation matrix of \( \vec{\varphi}^{(2)} \) is denoted as \( \mathbf{R}^{(2)} \).
   An additional random vector is defined as their sum:
   \[
   \vec{\varphi}^{\text{sum}} = \vec{\varphi}^{(1)} + \vec{\varphi}^{(2)}
   \]

Express the autocorrelation matrix of \( \vec{\varphi}^{\text{sum}} \), denoted as \( \mathbf{R}^{\text{sum}} \), in terms of \( \mathbf{R}^{(1)} \) and \( \mathbf{R}^{(2)} \). Show the mathematical developments leading to your result.
d. The random vector $\vec{\phi}$ is deteriorated by a linear degradation operator $H$ and then an additive noise vector $\vec{n}$, resulting in the degraded signal

$$\vec{\phi}_{deg} = H\vec{\phi} + \vec{n}$$

The noise $\vec{n}$ and the signal $\vec{\phi}$ are independent.

The signal autocorrelation matrix is denoted as $R_{\vec{\phi}}$, and the noise-vector autocorrelation matrix is denoted as $R_{\vec{n}}$.

- What is the autocorrelation matrix of the degraded signal $\vec{\phi}_{deg}$?
- Formulate the Wiener filter appropriate to the above problem.

**Question #2 (12 points)**

A Gaussian random variable $x \sim N(\mu_x, \sigma_x^2)$ is corrupted by an additive Gaussian noise sample $n \sim N(0, \sigma_n^2)$, which is statistically independent of $x$, resulting in the measured noisy variable $y = x + n$.

What is the probability density function of the random variable $y$? Provide a full mathematical proof.

Hint: recall that the probability density function (PDF) of a sum of two independent random variables is the convolution of their PDFs.

**Question #3 (13 points)**

The signal vector $\vec{x}$ is corrupted by an additive noise vector $\vec{n}$, so the measured noisy signal is $y = \vec{x} + \vec{n}$.

The observed noisy signal, $\vec{y}$, is linearly filtered to obtain the estimate:

$$\hat{\vec{x}} = H\vec{y}$$

where $H$ is the filter matrix.

Prove that for the optimal linear filter (in the minimal expected MSE sense), the error is orthogonal to the noisy measurements, i.e., show that $E[\vec{y}^T \vec{e}] = 0$,

where $\vec{e} \triangleq \vec{x} - \hat{\vec{x}}$ is the error signal.

Please note the following useful formulas, defined for two matrices $A$ and $B$:

$$\text{trace} \{ AB \} = \text{trace} \{ BA \}$$

$$\frac{\partial}{\partial A} \text{trace} \{ AB \} = \frac{\partial}{\partial A} \text{trace} \{ BA \} = B$$

$$\frac{\partial}{\partial A} \text{trace} \{ ABA^T \} = A(B + B^T)$$
Matlab Part (35 points) – Random Signals and Denoising using the Wiener Filter

General instructions:

- Figures should be appropriately titled.
- Your written report should be submitted with the dry part, whereas the code should be electronically submitted.

Consider again the class of one-dimensional discrete signals is defined above in Question #1 and note the specific definition given here below in red:

A signal is the column vector of \( N \) samples, where \( N \) is an even number, of the form:

\[
\overline{\varphi} = [M, \ldots, M, M + L, M, \ldots, M, M + L, M, \ldots, M]^T
\]

i.e., all the vector components have the value \( M \) except for the components in the \( K \) and the \( \left( K + \frac{N}{2} \right) \) coordinates that have the value \( M + L \). Note that \( K \in \{1, \ldots, \frac{N}{2}\} \).

The vector components are indexed starting at 1, i.e., the vector can be generally formulated as \( \overline{\varphi} = [\varphi_1, \ldots, \varphi_N]^T \).

\( M, L, \) and \( K \) are independent random variables:

- \( K \) is a uniform random variable over the integers \( \{1, \ldots, \frac{N}{2}\} \).
- \( M \) obeys \( E\{M\} = 0 \) and \( E\{M^2\} = c \)
- \( L \) obeys \( E\{L\} = 0 \) and \( E\{L^2\} = \frac{N}{2}(1 - c) \)

where \( 0 < c < 1 \) is a (deterministic) constant.

Consider signals of length \( N = 64 \) samples.
Moreover, the random variables \( M \) and \( L \) have here Gaussian distributions that obey the second-order statistics of the class for a parameter = 0.8.

a. Produce a large (and sufficient) amount of realizations of the class defined above (note that each signal realization relies on realizations of \( K, M \) and \( L \)).

Use these realizations to calculate the empirical approximation of the mean signal and the autocorrelation matrix of the class.

- Present the empirical mean of the class (using the ‘plot’ command), and show the empirically estimated autocorrelation matrix (using the ‘imagesc’ command).
- How well do your empirical results approximate the analytical results obtained in Question #1?
- What is the number of realizations needed for obtaining a good empirical approximation of the second-order statistics?
b. Signals of the above class are deteriorated by an additive white Gaussian noise:

\[ \overline{\varphi}_{\text{noisy}} = \overline{\varphi} + \overline{n} \]

where

\( \overline{n} \) is an additive noise vector, considered as a realization of an \( N \)-length random vector, having i.i.d components (with variance \( \sigma_n^2 \)) that follow

\[ \mu_n \triangleq E\overline{n} = 0 \quad \text{and} \quad R_n \triangleq E\overline{n}\overline{n}^T = \sigma_n^2 I \]

\( \overline{\varphi}_{\text{noisy}} \) is the given degraded signal (a \( N \)-length column vector).

Consider a noise variance of \( \sigma_n^2 = 1 \).

- Construct in Matlab the Wiener filter for denoising the above defined noisy signals.
  Show the filter matrix using the ‘imagesc’ command and explain the filter structure.
- Produce a large (and sufficient) amount of realizations of the class defined above, each should be deteriorated by a (different) realization of the noise vector defined above.
  Note that the clean signal should be kept for the purpose of computing the MSE of the denoised signal. However, the denoising task should be applied with respect to the noisy signal only.
- Use the Wiener filter constructed above to denoise the realizations of the noisy signals.
  - Plot several examples of denoised signals with respect to their clean and the noisy versions.
  - For each denoised realization, compute the MSE with respect to the clean version of the same realization.
    Average the MSE for all the realizations and write this number. This is the empirical approximation of the expected MSE of denoising using the Wiener filter.

c. Repeat subsection (b) for a noise variance of \( \sigma_n^2 = 5 \). Explain the differences in the obtained results.