Tutorial 7

The Continuous Fourier Transform
Dirac’s Delta Function
Convolution

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The Delta Function

- Definition (the continuous version \ Dirac’s delta)
  \( \delta(t) \) is a generalized function, defined using the integral:

  \[
  \int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \phi(0)
  \]

  where \( \phi(t) \) is a function that is continuous around \( t=0 \).

  - \( \delta(t) \) is an ideal description of an infinitesimally-narrow pulse around the origin, that defines area of 1.

    Since \( \delta(t) \) can be arbitrary shaped, it is **symbolically** represented as:

    (note that it is still a continuous function)

- Basic property: \( \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \)

  - The proof is by choosing \( \phi(t) \equiv 1 \).
The Delta Function: Properties

- Shifting: for any $\phi(t)$, which is continuous around $t_0$, it is held that
  \[ \int_{-\infty}^{\infty} \delta(t-t_0)\phi(t)\,dt = \phi(t_0) \]

- For any $t_0 \in \mathbb{R}$ it is held that
  \[ \int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1 \]

- The relation to the step function $u(t)$:
  \[ \int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 
  1, & t > 0 \\
  0, & t < 0 
\end{cases} \triangleq u(t) \rightarrow \delta(t) = \frac{d}{dt}u(t) \]
The Continuous Fourier Transform (1D)

• Defined as a transform of a continuous signal to its continuous transform-domain representation:

\[ \hat{\phi}(\xi) = F\{\phi(t)\} \triangleq \int_{-\infty}^{\infty} \phi(t) e^{-j2\pi \xi t} \, dt \]

where \( j = \sqrt{-1} \)

• The inverse transform:

\[ \phi(t) = F^{-1}\{\hat{\phi}(\xi)\} \triangleq \int_{-\infty}^{\infty} \hat{\phi}(\xi) e^{j2\pi \xi t} \, d\xi \]

– Note that the above integrals may, in general, result in a complex-valued functions.

• A sufficient (however, not necessary) condition for existence of the transform-domain representation:

\[ \int_{-\infty}^{\infty} |\phi(t)| \, dt < \infty \quad \text{(absolute integrability)} \]
The Continuous Fourier Transform (1D)

a. Calculate the Fourier transform of $\delta(t)$.

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi\xi t} dt = e^{-j2\pi\xi \cdot 0} = 1$$

$$\Rightarrow F\{\delta(t)\}_{(\xi)} \equiv 1$$

b. What is the inverse transform of $\delta(\xi)$?

$$F^{-1}\{\delta(\xi)\} = \int_{-\infty}^{\infty} \delta(\xi)e^{j2\pi t \xi} d\xi = e^{j2\pi t \cdot 0} = 1$$

$$\Rightarrow F^{-1}\{\delta(\xi)\}_{(t)} \equiv 1$$
The Continuous Fourier Transform (1D): Properties

Time-Frequency Duality:

If \( F\{\varphi(t)\} = \hat{\varphi}(\xi) \) then \( F\{\hat{\varphi}(t)\} = \varphi(-\xi) \).

Proof:

\[
\varphi(t) = \int_{-\infty}^{\infty} \hat{\varphi}(\xi)e^{j2\pi\xi t} \, d\xi
\]

\[
\rightarrow \quad \varphi(-t) = \int_{-\infty}^{\infty} \hat{\varphi}(\xi)e^{-j2\pi\xi t} \, d\xi
\]

Switching the roles of \( t \) and \( \xi \) gives:

\[
\varphi(-\xi) = \int_{-\infty}^{\infty} \hat{\varphi}(t)e^{-j2\pi\xi t} \, dt = F\{\hat{\varphi}(t)\}
\]
The Continuous Fourier Transform (1D): Properties

- **Linearity:** \( F\{a \cdot f(t) + b \cdot g(t)\} = a \cdot F\{f(t)\} + b \cdot F\{g(t)\} \)

- **Scaling:** \( F\{\phi(at)\} = \frac{1}{|a|} \cdot \hat{\phi}\left(\frac{\xi}{a}\right) \)

- **Shifting:** \( F\{\phi(t - t_0)\} = e^{-2\pi j t_0 \xi} \cdot \hat{\phi}(\xi) \)

- **Modulation:** \( F\{e^{2\pi j t_0 \xi} \phi(t)\} = \hat{\phi}(\xi - \xi_0) \)

- **Realness-Symmetry Duality:**
  If \( \phi(t) \) is real, then \( \hat{\phi}(-\xi) = \hat{\phi}^*(\xi) \) (conjugate-symmetric)

  => If \( \phi(t) \) is real and symmetric, then \( \hat{\phi}(\xi) \) is also a real function.
The Continuous Convolution

The convolution of two functions \( g, h \) is defined as

\[
(g * h)(t) = \int_{\tau=-\infty}^{\infty} g(\tau) h(t - \tau) d\tau
\]

- The convolution result is a function of \( t \).
- The integration variable is \( \tau \), hence, \( h(\cdot) \) is reflected and shifted by \( t \).

Let us intuitively demonstrate the convolution of:

So, for some \( t \in (-\infty, \infty) \), the functions in the integration in \((g * h)(t)\) are:

and the integration of \( g(\tau) \cdot h(t - \tau) \) is over the support-intersection region (see in blue):

Note that this is the continuous and non-cyclic version of convolution.
Fourier Transform and Convolution: 
The Continuous Case

- Fourier-transform property:
  Time-domain convolution transforms into frequency-domain multiplication

\[ F \{ g \ast h \}(\xi) = F \{ g(t) \}(\xi) \cdot F \{ h(t) \}(\xi) \]

Proof:

\[ F \{(g \ast h)(t)\}(\xi) = F \left\{ \int_{\tau=-\infty}^{\infty} g(\tau) h(t-\tau) \, d\tau \right\}(\xi) = \int_{\tau=-\infty}^{\infty} g(\tau) F \{ h(t-\tau) \}(\xi) \, d\tau \]

\[ = \int_{\tau=-\infty}^{\infty} g(\tau) \cdot e^{-j2\pi\xi\tau} F \{ h(t) \}(\xi) \, d\tau = F \{ h(t) \}(\xi) \cdot \int_{\tau=-\infty}^{\infty} g(\tau) \cdot e^{-j2\pi\xi\tau} \, d\tau = F \{ h(t) \}(\xi) \cdot F \{ g(t) \}(\xi) \]
The Continuous Fourier Transform (1D): Examples

Calculate the Fourier transform of the signal \( \varphi(t) = \cos(2\pi \xi_0 t) \).

Note that this signal is not absolutely-integrable, however, has a transform.

\[
e^{j2\pi \xi t} = \cos(2\pi \xi t) + j \sin(2\pi \xi t) \quad \quad e^{-j2\pi \xi t} = \cos(2\pi \xi t) - j \sin(2\pi \xi t)
\]

\[\Rightarrow \varphi(t) = \cos(2\pi \xi_0 t) = \frac{1}{2} \left[ e^{2\pi j \xi_0 t} + e^{-2\pi j \xi_0 t} \right] \]

We know that \( F\{\delta(t)\} = 1 \),
the time-frequency duality yields \( F\{1(t)\} = \delta(\xi) \),
and using the modulation property we get \( F\{e^{2\pi j \xi_0 t}\} = \delta(\xi - \xi_0) \).

Using the linearity property we can write:

\[
F\{\cos(2\pi \xi_0 t)\} = \frac{1}{2} \left[ F\{e^{2\pi j \xi_0 t}\} + F\{e^{-2\pi j \xi_0 t}\} \right] = \frac{1}{2} \left[ \delta(\xi - \xi_0) + \delta(\xi + \xi_0) \right]
\]

Similarly:

\[
F\{\sin(2\pi \xi_0 t)\} = \frac{1}{2j} \left[ \delta(\xi - \xi_0) - \delta(\xi + \xi_0) \right]
\]