Statistical Data Processing (236200)

Tutorial 6

The Discrete Fourier Transform (DFT)
The DFT Matrix

The DFT matrix of size $M \times M$ is defined as

$$[\text{DFT}] = \frac{1}{\sqrt{M}} \begin{bmatrix} (W^*)^{0.0} & \cdots & (W^*)^{0.(M-1)} \\ \vdots & \ddots & \vdots \\ (W^*)^{(M-1).0} & \cdots & (W^*)^{(M-1).(M-1)} \end{bmatrix}$$

where $W = e^{i \frac{2\pi}{M}}$

$$i = \sqrt{-1}$$

and $^{*}$ denotes complex conjugate.

The DFT matrix is symmetric and unitary, hence, its inverse is

$$[\text{DFT}]^* = \frac{1}{\sqrt{M}} \begin{bmatrix} W^{0.0} & \cdots & W^{0.(M-1)} \\ \vdots & \ddots & \vdots \\ W^{(M-1).0} & \cdots & W^{(M-1).(M-1)} \end{bmatrix}$$

i.e., $[\text{DFT}]^*[\text{DFT}] = [\text{DFT}][\text{DFT}]^* = I$
Representation of a Discrete Signal in the DFT Domain

Uniform sampling of the continuous signal $\varphi(t)$ provides us the discrete set of $M \equiv 2N + 1$ samples:

$$\varphi_{-N}, \varphi_{-(N-1)}, \ldots, \varphi_0, \ldots, \varphi_{N-1}, \varphi_N$$

that can be also arranged in a column vector as follows:

$$\varphi = \begin{bmatrix}
\varphi_0 \\
\varphi_1 \\
\vdots \\
\varphi_N \\
\varphi_{-N} \\
\varphi_{-(N-1)} \\
\vdots \\
\varphi_{-1}
\end{bmatrix}$$

Note the position of the negative-indexed samples.
Representation of a Discrete Signal in the DFT Domain

The representation of the discrete signal $\varphi$ is

$$\varphi^F = [\text{DFT}] \varphi$$

or in a more explicit form:

$$\begin{bmatrix}
\varphi_0^F \\
\varphi_1^F \\
\vdots \\
\varphi_N^F \\
\varphi_{-(N-1)}^F \\
\vdots \\
\varphi_{-1}^F 
\end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix}
(W^*)^{0\cdot0} & \cdots & (W^*)^{0\cdot(M-1)} \\
\vdots & \ddots & \vdots \\
(W^*)^{(M-1)\cdot0} & \cdots & (W^*)^{(M-1)\cdot(M-1)} 
\end{bmatrix} \begin{bmatrix}
\varphi_0 \\
\varphi_1 \\
\vdots \\
\varphi_N \\
\varphi_{-(N-1)} \\
\vdots \\
\varphi_{-1} 
\end{bmatrix}$$

Note that the negative index $-k$ (for $k = 1, \ldots, N$) can be considered also as the positive index $M - k$. 
Representation of a Discrete Signal in the DFT Domain

The DFT-domain representation is obtained via

\[ \phi^F = [\text{DFT}] \phi \]

Multiplying both sides of the equation by \([\text{DFT}]^*\), i.e.,

\[ [\text{DFT}]^* \phi^F = [\text{DFT}]^* [\text{DFT}] \phi \]

and, as the DFT matrix is unitary, we get

\[ \phi = [\text{DFT}]^* \phi^F \]

which is the inverse DFT procedure:
Given \( \phi^F \) it provides the signal-domain representation \( \phi \).
Consider the following discrete signal of $M$ samples:

For $n = 0, \ldots, M - 1$:

$$\varphi_n = \delta_{n,n_0} \triangleq \begin{cases} 1, & \text{for } n = n_0 \\ 0, & \text{otherwise} \end{cases}$$

where $n_0 \in \{0, \ldots, M - 1\}$

$\delta_{n,n_0}$ is also known as the Kronecker delta, here shifted to $n_0$.

The DFT of the above signal is

$$\varphi_F = \frac{1}{\sqrt{M}} \begin{bmatrix} (W^*)^{0\cdot0} & \cdots & (W^*)^{0\cdot(M-1)} \\ \vdots & \ddots & \vdots \\ (W^*)^{(M-1)\cdot0} & \cdots & (W^*)^{(M-1)\cdot(M-1)} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix} (W^*)^{0\cdot n_0} \\ (W^*)^{1\cdot n_0} \\ \vdots \\ (W^*)^{(M-1)\cdot n_0} \end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix} e^{-i\frac{2\pi}{M}0\cdot n_0} \\ e^{-i\frac{2\pi}{M}1\cdot n_0} \\ \vdots \\ e^{-i\frac{2\pi}{M}(M-1)\cdot n_0} \end{bmatrix}$$

Note the particular case of $n_0 = 0$. 

DFT Example #1: The Kronecker Delta
DFT Example #2: Cosine Signal

Consider the following discrete signal of \( M \) samples:

For \( n = 0, \ldots, M - 1 \):
\[
\varphi_n = \cos\left(\frac{2\pi k_0}{M} n\right)
\]

where \( k_0 \in \{0, \ldots, M - 1\} \)

Recall that
\[
\cos\left(\frac{2\pi k_0}{M} n\right) = \frac{1}{2} e^{i \frac{2\pi k_0}{M} n} + \frac{1}{2} e^{-i \frac{2\pi k_0}{M} n} = \frac{1}{2} (W^{k_0 n} + W^{-k_0 n})
\]

The \( k^{th} \) component of the DFT-domain representation of the above signal is

\[
\varphi_k^F = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} (W^*)^{k\cdot n} \varphi_n = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} W^{-k\cdot n} \frac{1}{2} (W^{k_0 n} + W^{-k_0 n}) =
\]

\[
= \frac{1}{\sqrt{M}} \left( \frac{1}{2} \sum_{n=0}^{M-1} W^{-k\cdot n} W^{k_0 n} + \frac{1}{2} \sum_{n=0}^{M-1} W^{-k\cdot n} W^{-k_0 n} \right) = \frac{1}{\sqrt{M}} \left( \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k-k_0)\cdot n} + \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k+k_0)\cdot n} \right)
\]
DFT Example #2: Cosine Signal

Let us examine the expression \( \sum_{n=0}^{M-1} W^{-(k-k_0)n} \):

For \( k = k_0 \):
\[
\sum_{n=0}^{M-1} W^{-(k-k_0)n} = \sum_{n=0}^{M-1} W^{-0.n} = \sum_{n=0}^{M-1} 1 = M
\]

For \( k \neq k_0 \):
\[
\sum_{n=0}^{M-1} W^{-(k-k_0)n} = \sum_{n=0}^{M-1} (W^{-(k-k_0)})^n = \frac{(W^{-(k-k_0)})^M - 1}{W^{-(k-k_0)} - 1}
\]

Recall that \( W = e^{i\frac{2\pi}{M}} \) [and that \( W^0, W^1, ..., W^{M-1} \) are the \( M \) roots (of order \( M \)) of unity].

Noting that \( (W^{-(k-k_0)})^M = (W^M)^{-(k-k_0)} = (e^{i\frac{2\pi}{M}M})^{-(k-k_0)} = (e^{i2\pi})^{-(k-k_0)} = 1 \) implies

For \( k \neq k_0 \):
\[
\sum_{n=0}^{M-1} W^{-(k-k_0)n} = 0
\]

\[
\sum_{n=0}^{M-1} W^{-(k-k_0)n} = M \cdot \delta_{k,k_0} \triangleq \begin{cases} M & \text{for } k = k_0 \\ 0 & \text{otherwise} \end{cases}
\]
DFT Example #2: Cosine Signal

Using the last result

\[
\sum_{n=0}^{M-1} W^{-(k-k_0)\cdot n} = M \cdot \delta_{k,k_0} \triangleq \begin{cases} M & \text{, for } k = k_0 \\ 0 & \text{, otherwise} \end{cases}
\]

The following development justifies the correspondence between the negative index \(-k_0\) and the index \(M - k_0\):

\[
\sum_{n=0}^{M-1} W^{-(k+k_0)\cdot n} = \sum_{n=0}^{M-1} W^{-(k+k_0-M)\cdot n} = \sum_{n=0}^{M-1} W^{-(k-(M-k_0))\cdot n} = M \cdot \delta_{k,M-k_0} \triangleq \begin{cases} M & \text{, for } k = M - k_0 \\ 0 & \text{, otherwise} \end{cases}
\]

We develop the expression for the \(k^{th}\) component of the DFT-domain representation of the cosine signal:

\[
\varphi_k^F = \frac{1}{\sqrt{M}} \left( \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k-k_0)\cdot n} + \frac{1}{2} \sum_{n=0}^{M-1} W^{-(k+k_0)\cdot n} \right) = \frac{1}{\sqrt{M}} \left( \frac{1}{2} M \cdot \delta_{k,k_0} + \frac{1}{2} M \cdot \delta_{k,M-k_0} \right) = \\
= \frac{\sqrt{M}}{2} \delta_{k,k_0} + \frac{\sqrt{M}}{2} \delta_{k,M-k_0}
\]
Image Enhancement in The DFT Domain

• We are given a noisy image of size 256×256:

\[ I_{noisy}[r,n] = I[r,n] + noise[r,n] \]

• The noise is **harmonic** and follows the formula:

\[ noise[r,n] = A_r \cdot \cos \left( 2\pi fn + \phi_r \right) \]

• \( f = \frac{1}{8 \text{ pixels}} \)

• The amplitude, \( A \), and the phase, \( \varphi \), are **random** and **independent for each line**.
Image Enhancement in The DFT Domain

\[ A_{100} = 22.37 \]
\[ \varphi_{100} = 1.325\text{rad} \]
Image Enhancement in The DFT Domain

Original Cameraman Image

Picture with noise
Image Enhancement in The DFT Domain
The Image-Domain Smoothing Alternative

Picture after average filter 1X4

Picture after average filter 1X6
Image Enhancement in The DFT Domain
Alternatives: Smoothing vs Median (8 pixels)

No noise but image is blurred

236200, CS Department, Technion
Image Enhancement in The DFT Domain

• DFT of the noise in line $r$

Recall that $M = 256$ and $f = \frac{1}{8}$, hence

$$noise_n^{(r)} = A_r \cos(2\pi fn + \phi_r) = A_r \cos\left(2\pi \frac{32}{M} n + \phi_r\right)$$

Then, since the signal is a shifted cosine function, its DFT is

$$DFT\{noise^{(r)}\}_k = \begin{cases} \frac{\sqrt{M}}{2} A_r e^{i\phi_r}, & k = 32, 224 \\ 0, & \text{else} \end{cases}$$

Here we would like to handle frequencies 32 and 224 (recall that 224 can also be considered as -32).
Image Enhancement in The DFT Domain

Spectrum of line 100 in picture with noise. n=32,224 marked with *

Noisy signal in DFT domain

Spectrum of line 100 After Notch Filtering. n=32,224 marked with *

Filtered signal in DFT domain

Notch Filter: Attenuate Specific Frequencies
Image Enhancement in The DFT Domain

- The noise was significantly removed.
- Original image was not fully restored
  - We cannot restore the attenuated frequencies
Image Enhancement in The DFT Domain

- Notch filter
- Smoothing filter of 8 pixels
Image Enhancement in The DFT Domain Implementation

• Filter in freq. domain:
\[
\text{Filter} = \text{ones}(1,256); \\
\text{Filter}(32+1)=0; \\
\text{Filter}(224+1)=0;
\]

• Filtration:
\[
\text{For } k=1:\text{size}(I,1), \\
Y = \text{fft}(I(k,:)) \ast \text{Filter}; \\
I(k,:)=\text{ifft}(Y);
\]
end
DFT Example #3: Periodic Delta Signal

Consider the following discrete signal of \( N \) samples:

For \( n = 0, \ldots, N - 1 \):

\[
\varphi_n = \begin{cases} 
1 & \text{for } n = 0, T, \ldots, (c - 1)T \\
0 & \text{otherwise}
\end{cases}
\]

where \( N = cT \) for some positive integer \( c \).

What is the DFT of \( \varphi \)?

**Solution:**

Using the definition of Kronecker’s delta

\[
\delta_{n,n_0} = \begin{cases} 
1 & , \text{for } n = n_0 \\
0 & , \text{otherwise}
\end{cases}
\]

we can write the signal as

\[
\varphi_n = \sum_{l=0}^{c-1} \delta_{n,Tl}
\]

where \( T \) and \( c \) were defined in the question.
DFT Example #3: Periodic Delta Signal

Recall the definition of the $N^{th}$ order root of the unity: $W_N = e^{\frac{i2\pi}{N}}$.

Then the $k^{th}$ DFT coefficient is

$$
\varphi_k^F = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (W_N^*)^{k \cdot n} \varphi_n = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left( (W_N^*)^{k \cdot n} \sum_{l=0}^{c-1} \delta_{n,Tl} \right) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \sum_{l=0}^{c-1} \delta_{n,Tl} \cdot (W_N^*)^{k \cdot n}
$$

$$= \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_N^*)^{k \cdot Tl}
$$

Since $N = cT$ we get that $$(W_N^*)^{k \cdot Tl} = \left(e^{-\frac{i2\pi}{N}}\right)^{k \cdot Tl} = e^{-\frac{i2\pi}{N} \cdot k \cdot Tl} = e^{-\frac{i2\pi}{c} \cdot kl} = (W_c^*)^{kl}$$

and therefore

$$\varphi_k^F = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_N^*)^{k \cdot Tl} = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_c^*)^{kl}$$
DFT Example #3: Periodic Delta Signal

For \( k = 0, c, ..., (T - 1)c \) we get that

\[
\frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_c^*)^{kl} = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (W_c^*)^k = \frac{1}{\sqrt{N}} \sum_{l=0}^{c-1} (1)^l = \frac{1}{\sqrt{N}} \cdot \frac{N}{T} = \frac{\sqrt{N}}{T}
\]

For \( k \neq 0, c, ..., (T - 1)c \) we calculate the sum of the geometric series as

\[
\sum_{l=0}^{c-1} (W_c^*)^{kl} = \frac{1 - (W_c^*)^k}{1 - (W_c^*)^k} = \frac{1 - ((W_c^*)^c)^k}{1 - (W_c^*)^k} = \frac{1 - 1}{1 - (W_c^*)^k} = 0
\]

where we used the fact that \((W_c^*)^c = (e^{-i2\pi/c})^c = e^{-i2\pi} = 1\).

To conclude, we got that

\[
\varphi_k^F = \begin{cases} 
\frac{\sqrt{N}}{T} & \text{for } k = 0, c, ..., (T - 1)c \\
0 & \text{otherwise}
\end{cases}
\]