Statistical Signal Processing  (236200)

Tutorial 4

Bit Allocation
Part I: Bit-Allocation for 2D Signals
Our 2D signal (image) is defined as a function \( \varphi(x, y) \), defined over the unit square.

\[
\varphi : [0,1] \times [0,1] \rightarrow \mathbb{R}
\]
Sampling

The sampling procedure includes:

Horizontal sampling: \( N_x \) samples in each row \( \Rightarrow \) sampling interval \( \Delta_x = \frac{1}{N_x} \)

Vertical sampling: \( N_y \) samples in each column \( \Rightarrow \) sampling interval \( \Delta_y = \frac{1}{N_y} \)

Overall \( N_x N_y \) samples,
representing the sampling-areas (generalizing the intervals in the one-dimensional problem):

\[
\Delta_{ij} = \left[ \frac{i}{N_x}, \frac{i+1}{N_x} \right] \times \left[ \frac{j}{N_y}, \frac{j+1}{N_y} \right] \quad i = 0, ..., N_x - 1; \quad j = 0, ..., N_y - 1
\]

of size \( Area(\Delta_{ij}) = \Delta_x \Delta_y = \frac{1}{N_x N_y} \)
Sampling

What are the optimal sampling-coefficients (in the MSE sense)?

The MSE of the \((i, j)\) slice, \(\Delta_{ij}\), is

\[
MSE_{\Delta_{ij}}(\varphi_{ij}) = \frac{1}{Area(\Delta_{ij})} \iint_{\Delta_{ij}} \left[ \varphi(x, y) - \varphi_{ij} \right]^2 dx dy
\]

\[
= \frac{1}{Area(\Delta_{ij})} \iint_{\Delta_{ij}} \left[ \varphi^2(x, y) - 2\varphi_{ij}\varphi(x, y) + \varphi_{ij}^2 \right] dx dy
\]

\[
= \frac{1}{Area(\Delta_{ij})} \iint_{\Delta_{ij}} \varphi^2(x, y) dx dy - 2\varphi_{ij} \frac{1}{Area(\Delta_{ij})} \iint_{\Delta_{ij}} \varphi(x, y) dx dy + \varphi_{ij}^2
\]

Finding the optimal \(\varphi_{ij}\) by \(\frac{d}{d\varphi_{ij}} MSE_{\Delta_{ij}}(\varphi_{ij}) = 0\):

\[-2 \frac{1}{Area(\Delta_{ij})} \iint_{\Delta_{ij}} \varphi(x, y) dx dy + 2\varphi_{ij} = 0\]

\[
\Rightarrow \varphi_{ij}^{opt} = \frac{1}{Area(\Delta_{ij})} \iint_{\Delta_{ij}} \varphi(x, y) dx dy
\]

The optimal sampling-coefficient is the average value of the sampling area.
Sampling

The minimal sampling error is given by plugging $\varphi_{ij}^{opt}$ in the MSE expression:

$$MSE_{\Delta_{ij}}^{Sampling} (\varphi_{ij}^{opt}) = \frac{1}{\text{Area} (\Delta_{ij})} \iint_{\Delta_{ij}} \varphi^2 (x, y) \, dxdy - \left( \frac{1}{\text{Area} (\Delta_{ij})} \iint_{\Delta_{ij}} \varphi (x, y) \, dxdy \right)^2$$
Quantization

• Our continuous signal is a real-valued function:

\[ \varphi(x, y) \in [\varphi_L, \varphi_H] \quad \text{where} \quad (x, y) \in [0,1] \times [0,1] \]

• We assume that the signal values are uniformly-distributed in the range \([\varphi_L, \varphi_H]\), i.e., their probability-density-function is

\[ p(z) = \frac{1}{\varphi_H - \varphi_L} \quad \text{for} \quad z \in [\varphi_L, \varphi_H] \]

• Hence, the expected squared-error of the quantization for \(b\) bits is the same as in the 1D case:

\[
E\left\{ \varepsilon_Q^2 \right\} = \int_{\varphi_L}^{\varphi_H} (z - Q(z))^2 \, p(z) \, dx = \sum_{i=1}^{2^b} \int_{d_{i-1}}^{d_i} (z - Q(z))^2 \, p(z) \, dx = \frac{\Delta_q^2}{12}
\]

where we considered a uniform quantizer with interval size \(\Delta_q = \frac{\varphi_H - \varphi_L}{2^b}\).
Sampling and Quantization

Let us define the following family of functions:

\[ 1_{\Delta_{ij}}(x, y) = \begin{cases} 
1 & (x, y) \in [(i-1)\Delta_x, i\Delta_x] \times [(j-1)\Delta_y, j\Delta_y] \\
0 & \text{else} 
\end{cases} \]

for \( i = 1, \ldots, N_x \) and \( j = 1, \ldots, N_y \)

Then, the signal reconstructed from sampling is

\[ \phi^S(x, y) = \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \phi_{ij}^{opt} \cdot 1_{\Delta_{ij}}(x, y) \]

and after quantization of sampling-coefficients we get

\[ \phi^Q(x, y) = \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \phi_{ij}^{opt,Q} \cdot 1_{\Delta_{ij}}(x, y) \]

The bit-budget constraint

Total number of samples: \( N = N_x N_y \)

Bits per sample (after quantization): \( b \)

The total bit budget is \( B \), and the constraint is expressed as \( B = N_x N_y b \)
Sampling and Quantization

The digitization (sampling+quantization) MSE (note the implicit normalization in the total area size of 1):

\[
MSE(N_x, N_y, b) = \int_{\xi=0}^{1} \int_{\eta=0}^{1} [\varphi(\xi, \eta) - \varphi^Q(\xi, \eta)]^2 \, d\xi \, d\eta = \int_{\xi=0}^{1} \int_{\eta=0}^{1} \varphi(\xi, \eta) - \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \varphi^Q \cdot 1_{\Delta_y}(\xi, \eta) \, d\xi \, d\eta
\]

\[
= \int_{\xi=0}^{1} \int_{\eta=0}^{1} \varphi^2(\xi, \eta) \, d\xi \, d\eta - 2 \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \varphi^Q \int_{\xi=0}^{1} \int_{\eta=0}^{1} \varphi(\xi, \eta) \cdot 1_{\Delta_y}(\xi, \eta) \, d\xi \, d\eta + \int_{\xi=0}^{1} \int_{\eta=0}^{1} \left( \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \varphi^Q \cdot 1_{\Delta_y}(\xi, \eta) \right)^2 \, d\xi \, d\eta
\]

\[
= Area(\Delta_{ij}) \cdot \varphi^Q = \frac{1}{N_x N_y} \varphi^Q
\]

\[
= Area(\Delta_{ij}) \cdot \varphi^Q = \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\varphi^Q)^2
\]

\[
= \int_{\xi=0}^{1} \int_{\eta=0}^{1} \varphi^2(\xi, \eta) \, d\xi \, d\eta - \frac{2}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \varphi^Q \cdot \varphi^Q + \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\varphi^Q)^2
\]

\[
= \int_{\xi=0}^{1} \int_{\eta=0}^{1} \varphi^2(\xi, \eta) \, d\xi \, d\eta + \frac{1}{N_x N_y} \left( \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\varphi^Q)^2 \right) - \frac{2}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \varphi^Q \cdot \varphi^Q + \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\varphi^Q)^2
\]

\[
= \int_{\xi=0}^{1} \int_{\eta=0}^{1} \varphi^2(\xi, \eta) \, d\xi \, d\eta + \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\varphi^Q - \varphi^Q)^2 - \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\varphi^Q)^2
\]

\[
= \int_{\xi=0}^{1} \int_{\eta=0}^{1} \varphi^2(\xi, \eta) \, d\xi \, d\eta - \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\varphi^Q)^2 + \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\varphi^Q - \varphi^Q)^2
\]

sampling error quantization error
Let us assume that the sampling regions are small enough such that we can approximate \( \varphi(x, y) \) over the \((i, j)\) sampling area, \( \Delta_{ij} \), using a first-order Taylor approximation:

for \((x, y) \in \Delta_{ij} :\)

\[
\varphi(x, y) \approx \varphi(x_c, y_c) + \varphi'_x(x_c, y_c) \cdot (x - x_c) + \varphi'_y(x_c, y_c) \cdot (y - y_c)
\]

where \((x_c, y_c)\) is the center of \( \Delta_{ij} \).

We denote \( A = \varphi(x_c, y_c) \), \( B_x = \varphi'_x(x_c, y_c) \) and \( B_y = \varphi'_y(x_c, y_c) \), so \( \varphi(x, y) \approx A + B_x \cdot (x - x_c) + B_y \cdot (y - y_c) \)

and approximate the average energy in \( \Delta_{ij} :\)

\[
\frac{1}{\text{Area}(\Delta_{ij})} \int_{x=i\Delta_x}^{x=(i+1)\Delta_x} \int_{y=j\Delta_y}^{y=(j+1)\Delta_y} \varphi^2(x, y) \, dx \, dy \approx \frac{1}{\Delta_x \Delta_y} \int_{x=x_c-\Delta_x/2}^{x=x_c+\Delta_x/2} \int_{y=y_c-\Delta_y/2}^{y=y_c+\Delta_y/2} \left[ A + B_x \cdot (x - x_c) + B_y \cdot (y - y_c) \right]^2 \, dx \, dy =
\]

(variable substitution is applied here)

\[
= \frac{1}{\Delta_x \Delta_y} \int_{\tilde{x}=-\Delta_x/2}^{\tilde{x}=\Delta_x/2} \int_{\tilde{y}=-\Delta_y/2}^{\tilde{y}=\Delta_y/2} \left[ A^2 + A B_x \tilde{x}^2 + A B_y \tilde{y}^2 + 2 A B_x \tilde{x} \tilde{y} + 2 A B_y \tilde{y}^2 \right] \, d\tilde{x} \, d\tilde{y} =
\]

\[
= \frac{1}{\Delta_x \Delta_y} \int_{\tilde{x}=-\Delta_x/2}^{\tilde{x}=\Delta_x/2} \int_{\tilde{y}=-\Delta_y/2}^{\tilde{y}=\Delta_y/2} A^2 \, d\tilde{x} \, d\tilde{y} + \int_{\tilde{x}=-\Delta_x/2}^{\tilde{x}=\Delta_x/2} \int_{\tilde{y}=-\Delta_y/2}^{\tilde{y}=\Delta_y/2} B_x^2 \tilde{x}^2 \, d\tilde{x} \, d\tilde{y} + \int_{\tilde{x}=-\Delta_x/2}^{\tilde{x}=\Delta_x/2} \int_{\tilde{y}=-\Delta_y/2}^{\tilde{y}=\Delta_y/2} B_y^2 \tilde{y}^2 \, d\tilde{x} \, d\tilde{y} =
\]

\[
= \frac{1}{\Delta_x \Delta_y} \left[ A^2 \Delta_x \Delta_y + B_x^2 \Delta_y \frac{\Delta_x^3}{12} + B_y^2 \Delta_x \frac{\Delta_y^3}{12} \right] = A^2 + B_x^2 \frac{\Delta_x^2}{12} + B_y^2 \frac{\Delta_y^2}{12} = \varphi^2(x_c, y_c) + \left( \varphi'_x(x_c, y_c) \right)^2 \frac{\Delta_x^2}{12} + \left( \varphi'_y(x_c, y_c) \right)^2 \frac{\Delta_y^2}{12}
\]

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\[ \text{MSE}_{\Delta_{ij}}^{\text{sampling}} = \frac{1}{\text{Area}(\Delta_{ij})} \iint_{\Delta_{ij}} \varphi^2(\xi, \eta) d\xi d\eta - \left( \varphi_{ij}^{\text{opt}} \right)^2 \]

\[ = \varphi^2(x_c, y_c) + \left( \varphi'_x(x_c, y_c) \right)^2 \frac{\Delta_x^2}{12} + \left( \varphi'_y(x_c, y_c) \right)^2 \frac{\Delta_y^2}{12} - \left( \varphi_{ij}^{\text{opt}} \right)^2 \]

We assume here that \( \varphi(\cdot, \cdot) \) is linear within the sampling areas.

Since the central-point of a linear function is also its average, we get:

\[ \varphi_{ij}^{\text{opt}} \approx \varphi(x_c, y_c) \]

This yields

\[ \text{MSE}_{\Delta_{ij}}^{\text{sampling}} \approx \left( \varphi'_x(x_c, y_c) \right)^2 \frac{\Delta_x^2}{12} + \left( \varphi'_y(x_c, y_c) \right)^2 \frac{\Delta_y^2}{12} \]
The above relates the MSE of the entire signal to the MSE values of the sampling intervals.
Formulating the Quantization Error

In the formula for the bit-allocation MSE, the quantization error is expressed as

$$\frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\phi_{ij}^{opt} - \phi_{ij}^{opt, Q})^2$$

which is an empirical averaging of the quantization errors for all the samples.

The squared errors of quantizing the samples, i.e.,

$$z_{ij}^Q = (\phi_{ij}^{opt} - \phi_{ij}^{opt, Q})^2$$, \quad i = 1, ..., N_x \quad j = 1, ..., N_y$$

are realizations of the random variable $\varepsilon_Q^2$.

Assuming that the number of samples is sufficiently large, the empirical average is close to the expected value of $\varepsilon_Q^2$:

$$\frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} (\phi_{ij}^{opt} - \phi_{ij}^{opt, Q})^2 \approx E\{z_{ij}^Q\}$$

We further assume that the samples are uniformly distributed in $[\varphi_L, \varphi_H]$, so for a $b$-bits uniform quantizer:

$$E\{z_{ij}^Q\} = \frac{1}{12} \cdot \frac{(\varphi_H - \varphi_L)^2}{2^{2b}}$$
Setting

\[ \text{MSE}_{\text{total}}^{\text{sampling}} = \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \text{MSE}_{\Delta_{ij}}^{\text{sampling}} \quad \text{and} \quad \text{MSE}_{\text{Quantization}} = \frac{1}{12} \left( \varphi_H - \varphi_L \right)^2 \]

in \( \text{MSE}_{\text{total}}^{\text{(N_x, N_y, b)}} = \text{MSE}_{\text{total}}^{\text{sampling}} + \text{MSE}_{\text{Quantization}} \)

yields

\[
\text{MSE}_{\text{total}}^{\text{(N_x, N_y, b)}} = \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \left[ \left( \varphi'_x (x_{ij}^c, y_{ij}^c) \right)^2 \frac{\Delta_x^2}{12} + \left( \varphi'_y (x_{ij}^c, y_{ij}^c) \right)^2 \frac{\Delta_y^2}{12} \right] + \frac{1}{12} \left( \varphi_H - \varphi_L \right)^2 2^{2b} = \\
= \frac{1}{12 N_x^2 N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \left( \varphi'_x (x_{ij}^c, y_{ij}^c) \right)^2 \Delta_x \Delta_y + \frac{1}{12 N_y^2 N_x} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \left( \varphi'_y (x_{ij}^c, y_{ij}^c) \right)^2 \Delta_x \Delta_y + \frac{1}{12} \left( \varphi_H - \varphi_L \right)^2 2^{2b}
\]

where \( (x_{ij}^c, y_{ij}^c) \) is the center of \( \Delta_{ij} \).

assuming \( \varphi(x, y) \) is linear in each region \( \Delta_{ij} \), the derivatives are constants and the sums here approximate integrals:

\[
= \frac{1}{12 N_x^2} \int_0^1 \int_0^1 \left( \varphi'_x (x, y) \right)^2 dxdy + \frac{1}{12 N_y^2} \int_0^1 \int_0^1 \left( \varphi'_y (x, y) \right)^2 dxdy + \frac{1}{12} \left( \varphi_H - \varphi_L \right)^2 2^{2b}
\]
Let us define
\[ \text{Energy}(\varphi'_x) = \int_0^1 \int_0^1 (\varphi'_x(x, y))^2 \, dx \, dy \]
\[ \text{Energy}(\varphi'_y) = \int_0^1 \int_0^1 (\varphi'_y(x, y))^2 \, dx \, dy \]

Then, the bit-allocation MSE is
\[ \text{MSE}^{\text{total}}(N_x, N_y, b) = \frac{1}{12N_x^2} \text{Energy}(\varphi'_x) + \frac{1}{12N_y^2} \text{Energy}(\varphi'_y) + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}} \]

Recall the constraint: \( B = N_x N_y b \)

The tradeoff here is among the horizontal resolution \( (N_x) \), vertical resolution \( (N_y) \), and quantization \( (b) \).

The corresponding optimization problem:
\[
\begin{align*}
\text{minimize} & \quad \text{MSE}^{\text{total}}(N_x, N_y, b) \\
\text{subject to} & \quad N_x N_y b = B
\end{align*}
\]
Part II: Bit-Allocation for 1D Signals
An Example
Bit-Allocation (1D): An Example

Consider the following signal for $t \in [0,1)$:

$$\varphi(t) = A \cdot \cos(2\pi \omega t)$$

where $A$ and $\omega$ are the signal’s amplitude and frequency parameters, respectively. We assume here that $\omega$ is a positive integer.

We would like to find the optimal bit-allocation for $\varphi(t)$. 
Bit-Allocation (1D): An Example

a. Express the derivative-energy \( Energy(\varphi') \) and the value-range \((\varphi_H - \varphi_L)\) of \( \varphi(t) \), as required for the bit-allocation optimization.

**Solution:**

Here \( \varphi_H = A \) and \( \varphi_L = -A \), and the derivative energy is calculated as

\[
Energy(\varphi') = \int_0^1 (\varphi'(t))^2 \, dt = \int_0^1 (-2\pi \omega A \cdot \sin(2\pi \omega t))^2 \, dt
\]

\[
= 4\pi^2 \omega^2 A^2 \int_0^1 \sin^2(2\pi \omega t) \, dt = 4\pi^2 \omega^2 A^2 \int_0^1 \frac{1}{2} (1 - \cos(4\pi \omega t)) \, dt
\]

\[
= 2\pi^2 \omega^2 A^2 \left( 1 - \frac{\sin(4\pi \omega t)}{4\pi \omega} \right |_0^1 ) = 2\pi^2 \omega^2 A^2
\]

\[= 0 \quad \text{(since } \omega \text{ is a positive integer)} \]
b. Find the optimal number of samples ($N_t$) and quantization bits ($b$) under the constraint of overall bit-budget $B$.

i. Formulate the bit-allocation optimization problem.

ii. Develop the problem to optimization on a single variable (i.e., $b$ or $N_t$), and formulate the mathematical expression (equation) that defines optimality.

**Solution:**

i. The bit-allocation optimization is

$$
\min_{N_t,b} \frac{1}{12N_t^2} \text{Energy}(\varphi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}}
$$

subject to $N_t b = B$
Bit-Allocation (1D): An Example

Solution:

We use the constraint to eliminate a variable: \( N_t = \frac{B}{b} \)

Then the optimization becomes

\[
\min_b \frac{b^2}{12B^2} \text{Energy}(\phi') + \frac{1}{12} \left( \phi_H - \phi_L \right)^2 \frac{2^{2b}}{2^{2b}}
\]

subject to \( b = \frac{B}{N_t} \)

the last optimization can also be written as (updating the constraints)

\[
\min_b \frac{b^2}{12B^2} \text{Energy}(\phi') + \frac{1}{12} \left( \phi_H - \phi_L \right)^2 \frac{2^{2b}}{2^{2b}}
\]

subject to \( 0 < b \leq B \)

\( N_t = \frac{B}{b} \)
Solution:

Let us optimize for $b$ while ignoring the constraints (that will be later checked to hold):

$$\min_b \frac{b^2}{12B^2} \text{Energy}(\phi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}}$$

Expression for optimality of $b$ is obtained by demanding the derivative equality to zero:

$$\frac{b}{6B^2} \text{Energy}(\phi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}} \cdot (-2^{1-2b} \ln(2)) = 0$$

that gives the optimality expression for $b$:

$$b \cdot 2^{2b} = \frac{(\varphi_H - \varphi_L)^2 \ln(2)}{\text{Energy}(\phi')} \cdot B^2$$

However, the last expression is analytically unsolvable.


**Bit-Allocation (1D): An Example**

**Section C**
You are given a total bit-budget of $B = 200$ bits. Compare (and explain) the values of the optimal bit-allocation parameters (i.e., $N_t$ and $b$) of the following signals:

$$
\varphi_1(t) = 5 \cdot \cos(2\pi t).
\varphi_2(t) = 5 \cdot \cos(20\pi t).
$$

Here you can use Matlab for numerically solving the optimality equation from section (b.ii). Useful Matlab functions here are `solve` and `lambertw`. Note that `solve` may return an expression that mathematically depends on the `lambertw` function, so, in turn, you can get a real valued result by appropriately applying the `lambertw` Matlab function.
Bit-Allocation (1D): An Example

**Solution:**
Numerical solution of the bit-allocation optimizations for $\varphi_1(t)$ and $\varphi_2(t)$ gives the corresponding solutions:

$$b_1^{opt} = 5.06 \quad b_2^{opt} = 2.30$$

We observe that the solutions hold the constraint $0 < b \leq B$ that was temporarily ignored.

For practical bit-allocation we need integer values of $b$. There is the closest integer from below, and the one from above. Note that the closer integer between the two does not necessarily provide the lower error. Accordingly, we need to evaluate the MSE for the two options:

Denoting the cost function as:

$$MSE_B(b) = \frac{b^2}{12B^2} Energy(\varphi') + \frac{1}{12} \frac{(\varphi_H - \varphi_L)^2}{2^{2b}}$$

For $\varphi_1(t)$: $MSE_B([b_1]) < MSE_B([b_1])$ and therefore $b_1^{opt,int} = 5$.

For $\varphi_2(t)$: $MSE_B([b_2]) < MSE_B([b_2])$ and therefore $b_2^{opt,int} = 2$.

The result that $b_1^{opt} > b_2^{opt}$ aligns with the fact that $\varphi_1(t)$ has lower frequency than $\varphi_2(t)$, hence for $\varphi_1(t)$ less samples are required allowing more bits for quantization.