Statistical Signal Processing  (236200)

Tutorial 3

Sampling
Signal Digitization: Motivation

“Real-world” signals:
- **Defined** over a *continuous* domain.
- Have **values** from a *continuous* and *possibly unbounded* range.

Digital representation:
- Discrete and finite domain and values
  - i.e., finite-length vectors of discrete values from a finite range.
- Required for
  - Processing
  - Storage
  - Transmission
  - Presentation
  
that involve computers or other digital devices/media.
Signal Sampling

Transfers a signal from a continuous domain to a discrete one:

Input:
A continuous function $\varphi(t)$ defined over the continuous interval $[0, 1]$

Output:
- A function $\varphi[n]$ defined on the discrete domain $n = 1, \ldots, N$
- Also formed as a finite length vector of $N$ elements:

\[
\begin{array}{c}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\vdots \\
\varphi_N
\end{array}
\]
Reconstruction from Sampling

Forming a continuous signal from the discrete (sampled) representation:

**Input:**
- A function \( \varphi[n] \) defined on the discrete domain \( n = 1, \ldots, N \)
- Also formed as a finite length vector of \( N \) elements:
  \[
  \varphi_1 \varphi_2 \varphi_3 \ldots \varphi_N
  \]

**Output:**
A continuous function \( \hat{\varphi}(t) \) defined over the continuous interval \([0,1]\)

The above is a simple reconstruction method of a constant value within each subinterval.
The Sampling Error

The sampling-reconstruction process introduces an error:

The error is the difference between the reconstruction and the continuous input-signal (defined for any $t \in [0,1]$): $e(t) = \varphi(t) - \hat{\varphi}(t)$

The error-level is affected by:
- The sampling procedure
- The reconstruction method
- The signal properties
Optimal Sampling

• Consider **uniform sampling** of $N$ samples in the area $[0,1)$:
  • All the sampling intervals have an equal (fixed) size of $\Delta = \frac{1}{N}$.
  • The sampling intervals are $\left[\frac{i-1}{N}, \frac{i}{N}\right]$ for $i = 1, \ldots, N$.

• Each sampling-interval is represented using a single value $\phi_i$.

• Optimal sampling-coefficients
  • Optimality is in terms of **objective error criterion** $C\{e(t)\}$
  • $C\{e(t)\}$ evaluates a given error signal using a non-negative scalar value, that should be minimized:
    $$\min_{\phi_1, \ldots, \phi_N} C\{e(t)\}$$

• Examples for $C\{\cdot\}$: **Mean-Squared-Error (MSE)**, **Mean-Absolute-Deviation (MAD)**
Optimal Sampling in the Mean-Squared-Error (MSE) Sense

\[
MSE_{\text{total}} = \frac{1}{N} \sum_{i=1}^{N} MSE_i
\]

where the MSE of the \(i^{th}\) interval is

\[
MSE_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} [\varphi(t) - \varphi_i]^2 dt
\]

Finding the optimal \(\varphi_i\) by requiring \(\frac{d}{d\varphi_i} MSE_i = 0\):

\[
\frac{d}{d\varphi_i} MSE_i = -\frac{2}{\Delta} \int_{(i-1)\Delta}^{i\Delta} [\varphi(t) - \varphi_i] dt = -\frac{2}{\Delta} \left[ \int_{(i-1)\Delta}^{i\Delta} \varphi(t) dt - \int_{(i-1)\Delta}^{i\Delta} \varphi_i dt \right] = -\frac{2}{\Delta} \left[ \int_{(i-1)\Delta}^{i\Delta} \varphi(t) dt - \Delta \cdot \varphi_i \right]
\]

\[\rightarrow \varphi_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \varphi(t) dt \]

The MSE-optimal coefficients are the interval averages.
Optimal Sampling in the Mean-Squared-Error (MSE) Sense

Exercise #1:

The signal \( \varphi(t) = \begin{cases} t, & \text{for } t \in [0,1] \\ 0, & \text{otherwise} \end{cases} \) is uniformly sampled to have \( N \) samples.

What are the optimal coefficients (in the MSE sense)?

The \( i^{th} \) interval is \( \left[ \frac{i-1}{N}, \frac{i}{N} \right) \)

\[
\varphi_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} t \, dt = \frac{1}{\Delta} \cdot \frac{t^2}{2} \bigg|_{(i-1)\Delta}^{i\Delta} = \frac{\Delta^2}{\Delta} \cdot \frac{i^2 - (i-1)^2}{2} = \Delta \cdot \left( i - \frac{1}{2} \right)
\]
Optimal Sampling in the Mean-Squared-Error (MSE) Sense

Exercise #1:

The signal \( \varphi(t) = \begin{cases} t, & \text{for } t \in [0,1] \\ 0, & \text{otherwise} \end{cases} \) is uniformly sampled to have \( N \) samples.

What are the MSE in the intervals?

\[
MSE_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} [t - \varphi_i]^2 dt = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \left[ t - \Delta \cdot \left( i - \frac{1}{2} \right) \right]^2 dt
\]

Change of integration variables, \( \tilde{t} = t - \left( i - \frac{1}{2} \right) \Delta \), leads to

\[
= \frac{1}{\Delta} \int_{\frac{\Delta}{2}}^{\frac{\Delta}{2}} \tilde{t}^2 d\tilde{t} = \frac{1}{\Delta} \cdot \frac{\tilde{t}^3}{3} \bigg|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{\Delta^2}{12}
\]
Optimal Sampling in the Mean-Absolute-Deviation (MAD) Sense

\[ \text{MAD}_{\text{total}} = \frac{1}{N} \sum_{i=1}^{N} \text{MAD}_i \]

where the MAD of the \( i^{th} \) interval is \( \text{MAD}_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} |\varphi(t) - \varphi_i| dt \)

Finding the optimal \( \varphi_i \):

\[ \frac{d}{d\varphi_i} \text{MAD}_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \frac{d}{d\varphi_i} |\varphi(t) - \varphi_i| dt = -\frac{1}{\Delta} \int_{t=(i-1)\Delta}^{i\Delta} \text{sign}(\varphi(t) - \varphi_i) dt \]

where we used \( \frac{d}{dx} |x| = \text{sign}(x) \), \( \text{sign}(x) \triangleq \begin{cases} 1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0 \end{cases} \)
Optimal Sampling in the Mean-Absolute-Deviation (MAD) Sense

Finding the optimal $\varphi_i$:

$$\frac{d}{d\varphi_i} MAD_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \frac{d}{d\varphi_i} |\varphi(t) - \varphi_i|dt = -\frac{1}{\Delta} \int_{t=(i-1)\Delta}^{i\Delta} \text{sign}(\varphi(t) - \varphi_i)dt$$

$$= -\frac{1}{\Delta} \int_{\varphi(t) > \varphi_i} 1dt - \frac{1}{\Delta} \int_{\varphi(t) < \varphi_i} (-1)dt - \frac{1}{\Delta} \int_{\varphi(t) = \varphi_i} 0 dt$$

Asserting zero-derivative gives, the optimality criterion for $\varphi_i$:

$$\int_{\varphi(t) > \varphi_i} dt = \int_{\varphi(t) < \varphi_i} dt$$

*The MAD-optimal coefficients can be interpreted as the interval medians.*
Optimal Sampling in the MAD Sense

Exercise #2:

The signal $\varphi(t) = \begin{cases} t , & \text{for } t \in [0,1] \\ 0 , & \text{otherwise} \end{cases}$ is uniformly sampled to have $N$ samples.

What are the optimal coefficients (in the MAD sense) ?

The $i^{th}$ interval is $[\Delta(i - 1), \Delta i)$

$$\varphi_i = \text{median}\{\varphi(t) | t \in [\Delta(i - 1), \Delta i]\} = \text{median}\{t \in [\Delta(i - 1), \Delta i]\} = \Delta \cdot \left(i - \frac{1}{2}\right)$$

$$MAD_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \left| t - \Delta \cdot \left(i - \frac{1}{2}\right) \right| dt = \frac{1}{\Delta} \int_{0}^{\frac{\Delta}{2}} \tilde{t} |\tilde{t}| d\tilde{t} = \frac{1}{\Delta} \cdot 2 \int_{0}^{\frac{\Delta^2}{8}} \tilde{t} d\tilde{t} = \frac{1}{\Delta} \cdot 2 \frac{\Delta^2}{8} = \frac{\Delta}{4}$$
Optimal Sampling: MSE vs. MAD

Exercise #3:

The signal \( \varphi(t) = \begin{cases} 
1 & , \ t \in \left[0, \frac{1}{4}\right) \\
\frac{4}{3}(1 - t), & , \ t \in \left[\frac{1}{4}, 1\right] \\
0 & , otherwise 
\end{cases} \)

is uniformly sampled to have \( N = 2 \) samples (\( \Delta = \frac{1}{2} \)).

What are the optimal coefficients (in the \textbf{MSE} sense) ?

\[
\varphi_{2}^{\text{MSE}} = \frac{1}{\Delta} \int_{\frac{1}{2}}^{1} \frac{4}{3}(1 - t)dt = 2 \cdot \frac{4}{3}t - \frac{4t^2}{3} \bigg|_{\frac{1}{2}}^{1} = \frac{1}{3}
\]

\[
\varphi_{1}^{\text{MSE}} = \frac{1}{\Delta} \int_{0}^{\frac{1}{4}} \varphi(t)dt = \frac{1}{\Delta} \int_{0}^{\frac{1}{4}} 1dt + \frac{1}{\Delta} \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{4}{3}(1 - t)dt = \frac{1}{2} + \frac{5}{12} = \frac{11}{12}
\]
Optimal Sampling: MSE vs. MAD

The signal $\varphi(t) = \begin{cases} 1 & , t \in \left[0, \frac{1}{4}\right) \\ \frac{4}{3}(1 - t), & t \in \left[\frac{1}{4}, 1\right] \\ 0, & otherwise \end{cases}$

is uniformly sampled to have $N = 2$ samples ($\Delta = \frac{1}{2}$).

What are the optimal coefficients (in the MAD sense)?

$$\varphi_{2}^{MAD} = \text{median}\left\{\varphi(t), t \in \left[\frac{1}{2}, 1\right]\right\} = \text{median}\left\{\frac{4}{3}(1 - t), t \in \left[\frac{1}{2}, 1\right]\right\} = \frac{1}{3}$$

$$\varphi_{1}^{MAD} = \text{median}\left\{\varphi(t), t \in \left[0, \frac{1}{2}\right]\right\}$$

$$= \text{median}\left\{1 \mid t \in \left[0, \frac{1}{4}\right]\right\} \cup \left\{\frac{4}{3}(1 - t) \mid t \in \left[\frac{1}{4}, \frac{1}{2}\right]\right\} = 1$$
Optimal Sampling: Another Example

Exercise #4:
Consider the signal \( \varphi(t) = at^2 \), \( t \in [0,1) \)
where \( a \) is a non-zero real-valued constant.

The signal is uniformly sampled to have \( N \) samples.

What are the optimal samples \( \varphi_1^{opt}, \ldots, \varphi_N^{opt} \) (in the MSE sense)?

Considering the MSE criterion, the optimal sample of the \( i^{th} \) interval is the interval average:

\[
\varphi_i^{opt} = \frac{1}{\Delta} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \varphi(t) dt
\]

where \( \Delta = \frac{1}{N} \).
Exercise #4 (cont.):
Consider the signal \( \varphi(t) = at^2 \), \( t \in [0,1) \)

What are the optimal samples \( \varphi_{\text{opt}}^1, \ldots, \varphi_{\text{opt}}^N \) (in the MSE sense)?

For the given quadratic signal the optimal sample becomes

\[
\varphi_{\text{opt}}^i = \frac{1}{\Delta} \int_{\frac{i}{N}}^{\frac{i-1}{N}} at^2 \, dt = \frac{1}{\Delta} \left. \frac{a}{3} t^3 \right|_{t=\frac{i-1}{N}}^{\frac{i}{N}} = \frac{a}{3\Delta} \left( \left( \frac{i}{N} \right)^3 - \left( \frac{i-1}{N} \right)^3 \right)
\]

\[
= \frac{a}{3\Delta N^3} (i^3 - (i-1)^3)
\]

\[
= \frac{a}{3\Delta N^3} (i^3 - (i^3 - 3i^2 + 3i - 1))
\]

\[
= \frac{a}{N^2} \left( i^2 - i + \frac{1}{3} \right)
\]
Optimal Sampling: Another Example

Exercise #4 (cont.):

What is the optimal reconstruction MSE? Provide an expression based on the general terms of $\varphi_1^{opt}, \ldots, \varphi_N^{opt}$ and $\varphi(t)$.

The optimal reconstruction MSE is

$$MSE = \int_0^1 \varphi^2(t) dt - \frac{1}{N} \sum_{i=1}^N (\varphi_i^{opt})^2$$

where

the total energy is

$$\int_0^1 \varphi^2(t) dt = \int_0^1 (at^2)^2 dt = \int_0^1 a^2 t^4 dt = \frac{a^2}{5} t^5 \bigg|_0^1 = \frac{a^2}{5}$$

and the average squared optimal-sample is

$$\frac{1}{N} \sum_{i=1}^N (\varphi_i^{opt})^2 = \frac{1}{N} \sum_{i=1}^N \left( \frac{a}{N^2} \left( i^2 - i + \frac{1}{3} \right) \right)^2 = \frac{a^2}{N^5} \sum_{i=1}^N \left( i^2 - i + \frac{1}{3} \right)^2$$

$$= \frac{a^2}{N^5} \sum_{i=1}^N \left( i^4 - 2i^3 + \frac{5}{3}i^2 - \frac{2}{3}i + \frac{1}{9} \right)$$
Optimal Sampling: Another Example

Exercise #4 (cont.):

Assume that $N$ is very large such that the sampling intervals are very small. Approximate the optimal reconstruction MSE only in terms of $a$ and $N$.

Following the lecture, when $N$ is large enough, the intervals are small enough such that the signal can be well approximated within each of them using a linear form. This leads to the sampling MSE formula:

$$MSE \approx \frac{1}{12N^2} \int_0^1 (\varphi'(t))^2 \, dt = \frac{1}{12N^2} \int_0^1 (2at)^2 \, dt$$

$$= \frac{4a^2}{12N^2} \int_0^1 t^2 \, dt = \frac{a^2}{3N^2} \cdot \frac{t^3}{3} \bigg|_0^1 = \frac{a^2}{9N^2}$$
Resampling

• Consider a discrete (already sampled) signal
• This signal can be sub-sampled to contain less samples
  • i.e., to be represented by a shorter vector.

This case is similar to the sampling of a continuous signal, however it has a discrete nature.
Resampling

$$MSE_{total} = \frac{1}{M} \sum_{i=1}^{M} MSE_i$$

where the MSE of the \(i^{th}\) interval (\(i = 1, \ldots, M\)) is

$$MSE_i = \frac{1}{D} \sum_{j=(i-1)D+1}^{iD} (\varphi_j - \tilde{\varphi}_i)^2$$

Similar to the continuous case, the optimal coefficient is the average of the “interval”:

$$\tilde{\varphi}_i = \frac{1}{D} \sum_{j=(i-1)D+1}^{iD} \varphi_j$$

A simple reconstruction procedure may be applied by

$$\hat{\varphi}_j = \tilde{\varphi}_k \quad \text{where} \quad k = \left\lfloor \frac{j-1}{D} \right\rfloor + 1 \quad \text{(for} \ j = 1, \ldots, N)$$