Geometric Modeling
Part II
Cubic Splines

- Standard spline input – set of points \( \{ P_i \} \) \( i = 0, n \)
  - No derivatives’ specified as input
- Interpolate by \( n \) cubic segments (4\( n \) DOF):
  - Force \( C^1 \) and \( C^2 \) continuity at points
  - Solve \( 4n \) linear equations in \( 4n \) unknowns

Interpolation (2\( n \) equations):
- \( C_i(0) = P_{i-1} \quad C_i(1) = P_i \quad i = 1, \ldots, n \)
- \( C^1 \) continuity constraints (\( n - 1 \) equations):
  - \( C'_i(1) = C'_{i+1}(0) \quad i = 1, \ldots, n - 1 \)
- \( C^2 \) continuity constraints (\( n - 1 \) equations):
  - \( C''_i(1) = C''_{i+1}(0) \quad i = 1, \ldots, n - 1 \)
Cubic Splines

- Have two degrees of freedom left (to reach $4n$ DOF)
- Options
  - Natural end conditions: $C_1''(0) = 0$, $C_n''(1) = 0$
  - Complete end conditions: $C_1'(0) = 0$, $C_n'(1) = 0$
  - Prescribed end conditions (derivatives available at the ends):
    $C_1'(0) = T_0$, $C_n'(1) = T_n$
  - Periodic end conditions
    $C_1'(0) = C_n'(1)$, $C_1''(0) = C_n''(1)$

- Question: What parts of $C(t)$ are affected as a result of a change in $P_i$?

  
  demo

Basis functions should be local
Parameterization

- The assumption $t \in [0,1]$ (uniform parameterization) is arbitrary
  - Implicitly implies same curve “length” for each segment
- Not natural if points are not equally spaced
- One alternative - chord-length parameterization:

Denote $d_i = \sqrt{(P_{i-1}^x - P_i^x)^2 + (P_{i-1}^y - P_i^y)^2}$
For the $i$'th segment : $t \in [0, d_i]$. 
Parameterization

Chord-length

Uniform

[0,1] [0,1] [0,8] [0,1] [0,1] [0,3] [0,1] [0,4]
Beziers Curves

- Bezier curve is an *approximation* of given control points
- Denote by $C(t): t \in [0, 1]$
- Bezier curve of degree $n$ is defined over $n+1$ control points $\{P_i\}_{i=0}^n$
De Casteljau Construction

Select $t \in [0,1]$ value.

For $i := 0$ to $n$ do $P_i^0(t) := P_i$;

For $j := 1$ to $n$ do

For $i := j$ to $n$ do

$P_i^j(t) := (1 - t)P_i^{j-1}(t) + tP_{i-1}^{j-1}(t)$;

$C(t) := P_n^n(t)$;

demo

$t = 1/3$

$C(1/3)$
Algebraic Form of Bezier Curves

Beziers curve for set of control points $\{P_i\}_{i=0,n}$:

$$C(t) = \sum_{i=0}^{n} P_i B_i^n(t) = \sum_{i=0}^{n} P_i \binom{n}{i} (1-t)^{n-i} t^i$$

$\{B_i^n(t)\}_{i=0,n}$ = Bernstein basis of polynomials of degree $n$

**Cubic case:**

$B_0^3(t) = (1-t)^3$
$B_1^3(t) = 3(1-t)^2 t$
$B_2^3(t) = 3(1-t)t^2$
$B_3^3(t) = t^3$
Algebraic Form of Bezier Curves

- Curve is linear combination of basis functions
- Curve is convex combination of control points

\[ C(t) = \sum_{i=0}^{n} P_i \binom{n}{i} (1-t)^{n-i} t^i \]

why?

\[ \sum_{i=0}^{n} B_i^n(t) = 1, \forall t \in [0, 1] \]
Properties of Bezier Curves

- $C(t)$ is polynomial of degree $n$
- $C(t) \in \text{CH}(P_0, \ldots, P_n)$
- $C(0) = P_0$ and $C(1) = P_n$
- $C'(t)$ is a Bezier curve of one degree less
- $C'(0) = n(P_1 - P_0)$ and $C'(1) = n(P_n - P_{n-1})$

- $C(t)$ is affine invariant and variation diminishing

\[ C(t) = \sum_{i=0}^{n} P_i \binom{n}{i} (1-t)^{n-i} t^i \]
Properties of Bezier Curves

Questions:

- What is the shape of Bezier curves whose control points lie on one line?

- How can one connect two Bezier curves with $C^0$ continuity? $C^1$? $C^2$?
Drawbacks of Bezier Curves

- Degree corresponds to number of control points
  - Global support: change in one control point affects the entire curve
  - For large sets of points – curve deviates far from the points

- Cannot represent conics exactly. Most noticeably circles
  - Can be resolved by introducing a more powerful representation of rational curves.
    - For example, a 90 degrees arc as a rational Bezier curve:
      \[
      C(t) = \frac{w_0 P_0 B_0^2(t) + w_1 P_1 B_1^2(t) + w_2 P_2 B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)}
      \]
      where \( \frac{w_0 w_2}{w_1^2} = 2 \).
Recap

Bernstein basis functions:

\[ B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i \]

\[ \forall t \in [0, 1], \ \forall i \ \ B_i^n(t) > 0, \ \sum_{i=0}^{n} B_i^n(t) = 1 \]

Bezier curve is \textit{linear} combination of basis functions:

\[ C(t) = \sum_{i=0}^{n} P_i B_i^n(t) \]

Bezier curve is \textit{convex} combination of control points (combination depends on \( t \)):

\[ C(t) = \sum_{i=0}^{n} P_i B_i^n(t) \]
B-Spline Curves

- Idea: Generate basis of functions with local support

- For each parameter value only a finite set of basis functions is non-zero
- The parametric domain is partitioned into sections at integer parameter values (called knots).

\[ C(t) = \sum_{i=0}^{n-1} P_i N_i(t) \]
Cubic B-Spline Basis

\[ C(t) = \sum_{i=0}^{n-1} P_i N_i(t), \quad t \in [3, n) \]

\[ N_i(t) = \begin{cases} 
    r^3 / 6 & r = t - i, \quad t \in [i, i+1) \\
    (-3r^3 + 3r^2 + 3r + 1) / 6 & r = t - (i+1), \quad t \in [i+1, i+2) \\
    (3r^3 - 6r^2 + 4) / 6 & r = t - (i+2), \quad t \in [i+2, i+3) \\
    (1-r)^3 / 6 & r = t - (i+3), \quad t \in [i+3, i+4) 
\end{cases} \]
Cubic B-Spline Basis
Cubic B-Spline Basis

- For any $t \in [3, n]$: 
  $$\sum_{i=0}^{n-1} N_i(t) = 1$$

- For any $t \in [3, n]$ at most four basis functions are non-zero.

- Any point on a cubic B-Spline is a convex combination of at most four control points.

$$C(t) = \sum_{i=0}^{n-1} P_i N_i(t)$$

$P_0$ to $P_3$ with corresponding $N_0(t)$ to $N_3(t)$ for $t \in [3, 4)$.
Boundary Conditions for Cubic B-Spline Curves

- B-Splines do not interpolate control points
  - in particular, the uniform cubic B-spline curves do not interpolate the end points of the curve.

- Ways to force endpoint interpolation:
  - Let \( P_0 = P_1 = P_2 \) (same for other end)
  - Add a new control point (same for other end) \( P_{-1} = 2P_0 - P_1 \) and a new basis function \( N_{-1}(t) \).
Local Control of B-spline Curves

Control point $P_i$ affects $C(t)$ only for $t \in (i, i+4)$
Properties of B-Spline Curves

- For \( n \) control points, \( C(t) \) is a piecewise polynomial of degree 3, defined over \( t \in [3, n) \)

\[
C(t) = \sum_{i=0}^{n-1} P_i N_i(t), \quad t \in [3, n)
\]

- \( C(t) \) is affine invariant and variation diminishing

- \( C(t) \) is *affine invariant* and *variation diminishing*

Questions:
- What is \( C(i) \) equal to?
- What is \( C'(i) \) equal to?
- What is the continuity of \( C(t) \)? Prove!
A curve is expressed as inner product of coefficients $P_i$ and basis functions

$$C(u) = \sum_{i=0}^{n} P_i B_i(u)$$

- Treat surface as a *curve of curves*. Also known as *tensor product surfaces*
- Assume $P_i$ is not constant, but are functions of a second, new parameter $v$:

$$P_i(v) = \sum_{j=0}^{m} Q_{ij} B_j(v)$$
From Curves to Surfaces (cont’d)

\[ C(u) = \sum_{i=0}^{n} P_i B_i(u) \]

\[ = \sum_{i=0}^{n} \left( \sum_{j=0}^{m} Q_{ij} B_j(v) \right) B_i(u) \]

\[ = \sum_{i=0}^{n} \sum_{j=0}^{m} Q_{ij} B_j(v) B_i(u) \]

\[ S(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} Q_{ij} B_j(v) B_i(u) \]
Isoparametric Curves
Surface Constructors

- Construction of the geometry is a first stage in any *image synthesis* process
- Use a set of high level, simple and intuitive, surface constructors:
  - Bilinear patch
  - Ruled surface
  - Boolean sum
  - Surface of revolution
  - Extrusion surface
  - Swept surface
Bilinear Patches

- Bilinear interpolation of 4 3D points - 2D analog of 1D linear interpolation between 2 points in the plane
- Given $P_{00}$, $P_{01}$, $P_{10}$, $P_{11}$ the bilinear surface for $u,v \in [0,1]$ is:

$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$
Questions:
What does an isoparametric curve of a bilinear patch look like?
Can you represent the bilinear patch as a Bezier surface?
When is a bilinear patch planar?
Given two curves \( a(t) \) and \( b(t) \), the corresponding ruled surface between them is:

\[
S(u,v) = v a(u) + (1-v)b(u)
\]

The corresponding points on \( a(u) \) and \( b(u) \) are connected by straight lines.

Questions:
- When is a ruled surface a bilinear patch?
- When is a bilinear patch a ruled surface?
Ruled Surfaces
גשר/mitrim
Boolean Sum

- Given four connected curves $\alpha_i \ i=1,2,3,4$, Boolean sum $S(u,v)$ fills the interior.

\[
P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}
\]

\[
S_1(u,v) = v\alpha_0(u) + (1-v)\alpha_2(u)
\]

\[
S_2(u,v) = u\alpha_1(v) + (1-u)\alpha_3(v)
\]

Then

\[
S(u,v) = S_1(u,v) + S_2(u,v) - P(u,v)
\]
Boolean Sum (cont’d)

- $S(u,v)$ interpolates the four $\alpha_i$ along its boundaries.

- For example, consider the $u = 0$ boundary:

\[
S(0, v) \\
= S_1(0, v) + S_2(0, v) - P(0, v) \\
= v\alpha_0(0) + (1-v)\alpha_2(0) + 0\alpha_1(v) + 1\alpha_3(v) - (1-v)P_{00} - vP_{01} \\
= vP_{01} + (1-v)P_{00} + \alpha_3(v) - (1-v)P_{00} - vP_{01} \\
= \alpha_3(v)
\]
Surface of Revolution

- Rotate a, usually planar, curve around an axis

- Consider curve
  \[ \beta(t) = (\beta_x(t), 0, \beta_z(t)) \]
  and let \( Z \) be the axis of revolution.

\[
\begin{align*}
  x(u, v) &= \beta_x(u) \cos(v), \\
  y(u, v) &= \beta_x(u) \sin(v), \\
  z(u, v) &= \beta_z(u).
\end{align*}
\]
Surfaces of Revolution
Extruded Surface

- Extrusion of a, usually planar, curve along a linear segment.

- Given curve $\beta(t)$ and vector $\vec{V}$

$$S(u, v) = \beta(u) + v\vec{V}$$
Swepted Surface

- Rigid motion of one (cross section) curve along another (axis) curve:

- The cross section may change as it is swept

Question: Is an extrusion a special case of a sweep? a surface of revolution?