Research Intro

Computer Aided geometric Design (CAGD)
Geometric Modeling (GM)

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Geometric Modeling
Geometric Modeling

- High end Geometric Modeling (GM) research focuses on the use of piecewise polynomial/rational (Bezier/B-spline) representations.
- In recent years, the area of GM, is being reshaped:
  - We have abilities to fabricate objects from heterogeneous materials.
  - We can 3D print highly complex (micro-structured) geometries.
  - We can establish tight links with analysis that employs spline basis functions.
  - Symbolic tools are entering GM.
Geometric Modeling

Herein, I will focus on research we do at the CGGC, CS, Technion:

- Volumetric modeling and micro-structures.
- Additive (3D printing) Manufacturing (AM).
- Subtractive (CNC) manufacturing (SM).
- Constraint solving.
- Intuitive GM for novice users.
- Art related: “Escher for Real”: 
Geometric Modeling

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- Art related: “Escher for Real” etc.
Volumetric Modeling and micro-structures
Introduction I

The current geometric design paradigm is with us for over four decades, almost unchanged:

- A B-spline based boundary representation (B-reps).
- B-reps were great when the manufactured artifacts were homogeneous.
- B-reps were never great with respect to analysis: Converting freeform B-reps to (piecewise linear) volumetric elements, for analysis, and back.
Introduction II

- We require a representation that will interact with ease and support all parties involved:
  - Analysis/optimization is typically volumetric.
  - Manufacturing is becoming volumetric.
  - Therefore, the design better be based on a geometric representation that is a volumetric representation (V-rep).
- We propose a V-rep that is based on Trimmed Trivariate B-splines.
- While aiming to manage backward B-rep compatibility.
Constructing Trivariate Geometry

End users of contemporary B-rep CAD systems are constructing geometry in well defined ways:

- **Primitives** (i.e. cones, spheres,…).
- **High level constructors** (extrusions, sweeps,…).
- **Boolean ops. over existing geometry** (have a closure).
- **Special ops. like composition**, offsets and fillets/blends.

The proposed V-rep better support the above as well as satisfy analysis & manufacturing needs.

We now consider some of the above, in V-reps.
High Level Geometry Constructors I

All high level surface constructors, in modern CAD systems, can also be extended to V-reps:

- From an Extrusion Surface to an Extruded Volume:

- From a Ruled Surface to a Ruled Volume:

\[
T(u, v, w) = S_1(u, v)(1 - w) + S_2(u, v)w
\]

\[
T(u, v, w) = S(u, v) + w\vec{v}
\]
High Level Geometry Constructors II

- From a Surface of Revolution to a Volume of Revolution:
- From a Sweep Surface to a Swept Volume:
- From a Boolean Sum Surface to a Volumetric Boolean Sum:
Boolean Operations

- The support of **Boolean operations** is a key for the **generality of geometry** we can synthesize.
- Boolean operations over V-reps add another dimension of complexity, compared to B-reps.
- Yet, we draw upon B-reps Boolean operation abilities to derive the boundary of the V-reps.
  - And complete the V-rep’s Booleans in the interior.
- A V-rep model (**V-model**) is a complex of (3-manifold) trimmed trivariate V-cells.
- A 3-manifold extended **half-edge structure**
Boolean Operations’ Examples I
Boolean Operations’ Examples II
Special Operations on V-reps I

Polynomial/rational V-reps are supported in:

- Fitting/Interpolation (skinning) over Surfaces.
- **Evaluation**: Positions, Derivatives & Integration
  - Both at single points or as fields.
- Subdivision, Refinements & Degree elevation.
- Point inclusion queries.
- Closest point queries to curve/surface/trivariate.
- Blending and (freeform) Deformation
- Precise point/curve projections/Contact analysis.
Special Operations on V-reps II

- Extensive mixed symbolic/numeric computations:
  - Addition & Subtraction of (multivariate) splines.
  - Multiplication, Cross & Dot product of splines.
  - Functional (symbolic) composition.
  - Zero set finding, etc.

Clearly not complete. I.e. V-reps are yet to support:
- Offsets
- Rounding/fillets
Precisely computed in the Bezier/B-spline basis (under some limitations), by tiling the domain of $T$ \(4 \times 4 \times 4\) times:
Surface-Trivariate Composition II

+ =
Surface-Trivariate Composition III
Curve-Trivariate Composition

Univariates’ volume covering can be derived to manage fiber placements (composites):
Trimmed Surface-Trivariate Composition

Can also be used over trimmed geometry:

Trimming information is simply propagated along.
Trivariate--Trivariate Composition
Iso-geometric analysis (IGA) in collaborations with Pablo Antolin (EPFL Lausanne), Annalisa Buffa (EPFL Lausanne and IMATI-CNR Pavia), Massimiliano Martinelli (IMATI-CNR Pavia); Giancarlo Sangalli (University of Pavia and IMATI-CNR Pavia)
Hierarchical Microstructures I

- Tiles need not be smooth or even continuous.
- Proper stitching between tiles (via linear constraints, preserving continuity for analysis)
  - Tile branch out faces(s) can be precisely glued to branch in face.
- Can create a highly complex hierarchical, fractal like, geometry.
  - Having self-similarity.
- Same representation in design and analysis, in the spirit of iso-geometric analysis.
Hierarchical Microstructures II – Heat Sinks

http://www.rapidled.com/small-single-led-heatsink
Iso-geometric analysis (IGA) in collaborations with Pablo Antolin (EPFL Lausanne), Annalisa Buffa (EPFL Lausanne and IMATI-CNR Pavia), Massimiliano Martinelli (IMATI-CNR Pavia); Giancarlo Sangalli (University of Pavia and IMATI-CNR Pavia)
Trivariate-Trivariate composition creates a Closure
Trivariate-Trivariate composition creates a Closure
Lower Order Approximations I

- Resulting composed geometry can be of high order.
- Iso-geometric analysis is computationally sensitive to the degrees of the freeform elements.
  - Typically quadratic and/or cubic degrees are preferred.
- One can lower order approximate the microstructure geometry using quadratic and cubic forms.
- We demonstrate using this wing microstructure:
Lower Order Approximations II

Tri-cubic fit

Tri-quadratic fit

Original

Tri-orders

16 x 6 x 11
Multiresolution Microstructures (MrMs)

Feasible due to the closure established by the degree reduction.
Slicing MrMs towards AM

Two levels of trivariate microstructures.
V-reps Toward AM – An Example

Trivariate--trivariate composition can also support interior properties such as multi-/graded-materials.
Un-trimming trivariates (V-reps) I

Untrimming the trimmed Bezier trivariates (into boundary-singular tensor products):
Un-trimming trivariates (V-reps) II

A complete (very recent) example.

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Managing the Interiors of V-reps

- Support and manage different interior attributes: scalar/vector/tensor properties, etc.

- These attributes can be additional fields alongside the 3D geometry, as additional (non-geometric) trivariates.

- Can come, for example, from analysis or optimization.

- But now certain V-cells will contain more than one trivariate.

- How to fuse/blend these fields then?

- What continuity is to be expected?
Additive Manufacturing (AM) (3D printing)
Some MS AM results

3D printing of porous/microstructure materials:

3D Printed courtesy of Stratasys Israel
V-reps Toward AM (additive manufacturing)

- Having a V-rep, one can analysis and (topologically) optimize the part and prescribe its interior (heterogeneous) material composition.
- Slicing V-reps by planes is clearly feasible (Solve for $T(u, v, w) = Z_0$).
- AM (and SM) is, in essence, about covering the geometry with univariates.
V-reps Toward AM – Example I

- Microstructures in trivariates
  - via symbolic composition.
Twisting colored fibers
V-reps Toward AM – Example III

The Utah teapot...
Non Slicing based AM via V-reps

- AM via slicing is virtually the only common approach.
- The slicing orientation is (almost) independent from the geometry.
- Has draw backs in the form of, for example:
  - Potentially inferior strength.
  - Potentially degraded surface finish.
- Having a V-rep, we can cover the volume of the object using general univariate (curves).
Volume $\Rightarrow$ Surfaces $\Rightarrow$ Curves
Non slicing AM – Example I

(a) 

(b)
Non slicing AM – Example II
Constraints Solving
Problem Statement

Consider the following set of $n$ rational constraints:

$$f_1(u_1, u_2, \ldots, u_m) = 0,$$

$$\vdots$$

$$f_n(u_1, u_2, \ldots, u_m) = 0,$$

In $\mathbb{R}^{m+1}$. $u = (u_1, u_2, \ldots, u_m)$.

We seek the simultaneous solution, $u^s \in \mathbb{R}^m$, such that $f_i(u^s) = 0$ for all $i = 1, \ldots, n$. 
The Solver

Having many ways (numeric/algebraic/etc.) to try and solve this problem we focus on a (geometric) subdivision based solver in the piecewise-polynomial/rational domains.

Well suited to use in geometry processing and in CAGD:

- Capable of handling zero-dimensional as well as non-zero-dimensional solution spaces.
- Up to certain tolerance limitations, capable of finding all real solutions.
The Bezier/NURBs Representation

Using properties of Bezier/B-spline functions one can derive a subdivision based solver for the problems in hand using:

- A domain reduction stage in a multidimensional space, using:
  - Bezier/B-spline subdivision/domain clipping,
  - Convex hull containment,
  - Preconditioning and single solution guarantee.
- A multivariate numeric improvement step (i.e. Newton-Raphson).
Let $C(t)$ be a $C^1$ planar rational parametric curve as before. The simultaneous solution of the two rational constraints of

\[ F_1 (r,t) = \langle C(t) - C(r), N(t) \rangle = 0, \]
\[ F_2 (r,t) = \langle C(t) - C(r), N(r) \rangle = 0, \]

prescribes the set of $(r, t)$ pairs of locations on $C(t)$ that defines the bi-tangents of $C(t)$.
The (10th) Apollonius problem

\[
\langle P - C_1(u), P - C_1(u) \rangle = \langle P - C_2(v), P - C_2(v) \rangle, \\
\langle P - C_1(u), P - C_1(u) \rangle = \langle P - C_3(w), P - C_3(w) \rangle, \\
\left\langle P - C_1(u), \frac{dC_1(u)}{du} \right\rangle = 0, \\
\left\langle P - C_2(v), \frac{dC_2(v)}{dv} \right\rangle = 0, \\
\left\langle P - C_3(w), \frac{dC_3(w)}{dw} \right\rangle = 0.
\]

Observation: \( C_i \) need not be circles!
Apollonius problem – General Curves
Kernel of Surfaces in $\mathbb{R}^3$

- The inflection points of curves could be generalized to the parabolic points in surfaces.
- Convex and concave (elliptic), parabolic, and saddle-like (hyperbolic):

Quantified by solving for $K = 0$, $K$ being the Gaussian curvature of $S$. 
Curve-Curve Bisector in $R^2$

\[ \langle B - C_1(t), B - C_1(t) \rangle = \langle B - C_2(r), B - C_2(r) \rangle, \]

\[ \langle B - C_1(t), \frac{dC_1(t)}{dt} \rangle = 0, \]

\[ \langle B - C_2(r), \frac{dC_2(r)}{dt} \rangle = 0. \]

**Question:** what about arrangements of planar curves?
Voronoi Regions in $R^2$

Compute the Voronoi Cell of $C_0(t)$ with respect to all other curves in the arrangement:

- Compute the bisector $B_i$ of $C_0(t)$ and all $C_i(r)$, $i > 0$.
- Compute the lower envelope of $\{B_i\}$.

These Voronoi cells are accurate to within machine precision!
Voronoi Regions in $R^2$ – More examples
Consider two planar $C^1$ parametric curves, $C(u)$ and $D(v)$.

We compute the double contact motion path of $D$ around $C$.

Parameterize both $C$ and $D$ twice as $C(u)$ & $C(r)$ and $D(v)$ & $D(s)$.
Motion Planning – Double Contact II

Assume $D(v)$ is fixed and $C(u)$, has three degrees of freedom to move ($x$ and $y$) and rotate ($\Theta$):

\[ R(\theta)[C(u)] + (x, y) = D(v), \]
\[ R(\theta)[C(r)] + (x, y) = D(s), \]
\[ R(\theta)[C'(u)] \parallel D'(v), \]
\[ R(\theta)[C'(r)] \parallel D'(s). \]

- First two eqns offer two constraints each (for $x$ and $y$)
- Last two eqns ensure tangency contact.
- 7 DOFS ($x$, $y$, $\Theta$, $u$, $v$, $r$, $s$) and 6 Eqns.
Motion Planning – Double Contact III

- Compute all valid triple contact points - 9 DOFS \( (x, y, \varTheta, u, v, r, s, a, b) \) & 9 Eqns:
  
  \[ R(\theta)[C(u)] + (x, y) = D(v), \]
  \[ R(\theta)[C(r)] + (x, y) = D(s), \]
  \[ R(\theta)[C(a)] + (x, y) = D(b), \]
  \[ R(\theta)[C'(u)] \parallel D'(v), \]
  \[ R(\theta)[C'(r)] \parallel D'(s), \]
  \[ R(\theta)[C'(a)] \parallel D'(b). \]

- Only to trace the double contact univariates, in between.

- Creates an arrangement of motion curves in \( (x, y, \varTheta) \) c-space.
Motion Planning – Double Contact IV
A C-space precise solution:
Motion Planning – Double Contact V

And possibly deformable robots:
Complexity Reduction: Constraints Decomposition

Jansen’s linkage

<table>
<thead>
<tr>
<th>Problem</th>
<th>Without decomposition</th>
<th>With decomposition</th>
<th>Speedup</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jansen’s linkage, without inequality constr.</td>
<td>19730 sec &gt;5.4 hours</td>
<td>61.2 sec</td>
<td>322.3</td>
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<td>2.7 sec</td>
<td>3530.2</td>
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Required Background

Work in GM requires

- Good background in Calculus.
- Background in Linear Algebra.
- Relation to Numerical Analysis
- Good programming abilities.

Relevant classes:

- The CAGD class 236716.
- The Computer Graphics class 234325 recommended.
- Differential geometry (math) recommended.