Lecture Notes on Acceleration of Linear Convergence by the Aitken Δ^2-Process

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1 Introduction

Definition 1.1 *Let $\{x_n\}$ be a real or complex sequence converging to α. We say that it* converges of order p $(p \geq 1)$ *if*

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = C, \quad 0 < C < \infty. \tag{1.1}$$

If $C = 0$, we say that $\{x_n\}$ converges of order at least p. *When $p = 1$, we need to have $C < 1$ for convergence to take place.* (*When $p = 1$, we also say that $\{x_n\}$* converges linearly. *When $p = 2$, we also say that $\{x_n\}$* converges quadratically. *And so on.*) *Here, the x_n and α are complex in general.*

What (1.1) means is

$$\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} \approx C \quad \forall n \text{ large.} \tag{1.2}$$

This can also be written as

$$|x_{n+1}-\alpha| \approx C|x_n-\alpha|^p \quad \forall n \text{ large.} \tag{1.3}$$

When $p = 1$, the error $|x_{n+1}-\alpha|$ is approximately C times the error $|x_n-\alpha|$. Thus, the rate of convergence depends on how far C is from 1. The closer C is to 1, the slower the convergence. We would like to design a method by which we can use $\{x_n\}$ to generate a new sequence of approximations $\{\hat{x}_n\}$ to α that will *converge faster than* $\{x_n\}$, in the sense that

$$\lim_{n\to\infty} \frac{\hat{x}_n - \alpha}{x_n - \alpha} = 0.$$

We also say, equivalently, that the method *accelerates the convergence of the sequence* $\{x_n\}$.

The simplest and most popular, and yet very effective, method that accelerates the convergence of linearly convergent sequences is the *Aitken* Δ^2*-process*, and we treat this method here. As will become clear soon, some of the meaningful results concerning this method can be proved using rather elementary calculus, and hence should be easy to teach in an introductory course in Numerical Analysis.

For an up-to-date account of the Δ^2-process, see the book by Sidi [1, Chapter 15, Section 5.3]. The material in these notes is taken partly from this book.

2 Aitken Δ^2-Process

Let us redefine linear convergence by removing the absolute value signs in (1.1) as follows:

Definition 2.1 *Let* $\{x_n\}$ *be a sequence converging to* α*. We say that it* converges linearly *if*

$$\lim_{n\to\infty} \frac{x_{n+1} - \alpha}{x_n - \alpha} = C, \quad 0 < |C| < 1. \tag{2.1}$$

Here, the x_n*,* α*, and* C *are complex in general.*

Starting with a sequence $\{x_n\}$ that converges linearly to some limit α, we now aim at obtaining another sequence $\{\hat{x}_n\}$ that, we hope, converges to α faster.

By (2.1), for all large n, we have

$$\frac{x_{n+1} - \alpha}{x_n - \alpha} \approx C,$$

Of course, we also have

$$\frac{x_{n+2} - \alpha}{x_{n+1} - \alpha} \approx C.$$

Therefore, we also have

$$\frac{x_{n+1} - \alpha}{x_n - \alpha} \approx \frac{x_{n+2} - \alpha}{x_{n+1} - \alpha}.$$

We can view this relation as an (approximate) equation, α being the only unknown; let us solve it for α. Cross multiplying, we obtain

$$(x_{n+1}-\alpha)^2 \approx (x_n-\alpha)(x_{n+2}-\alpha).$$

Expanding, we have

$$(x_n - 2x_{n+1} + x_{n+2})\alpha \approx x_n x_{n+2} - x_{n+1}^2.$$

Solving for α, we finally obtain

$$\alpha \approx \frac{x_n x_{n+2} - x_{n+1}^2}{x_n - 2x_{n+1} + x_{n+2}}. \tag{2.2}$$

Let us denote

$$\hat{x}_n = \frac{x_n x_{n+2} - x_{n+1}^2}{x_n - 2x_{n+1} + x_{n+2}}. \tag{2.3}$$

This process, by which we have generated the sequence $\{\hat{x}_n\}$ starting from the sequence $\{x_n\}$, is known as the Δ^2-*process of Aitken.* It is easy to verify that $\hat{x}_n$ can also be expressed as in

$$\hat{x}_n = x_n - \frac{(\Delta x_n)^2}{\Delta^2 x_n}, \tag{2.4}$$

$$\hat{x}_n = x_{n+1} - \frac{(\Delta x_n)(\Delta x_{n+1})}{\Delta^2 x_n}, \tag{2.5}$$

where we have defined

$$\Delta x_n = x_{n+1} - x_n \quad \text{and} \quad \Delta^2 x_n = \Delta(\Delta x_n) = x_{n+2} - 2x_{n+1} + x_n.$$ [1]

Note that (2.4) and (2.5) give numerically more stable ways of computing $\hat{x}_n$.

3 Acceleration of linear convergence: a general theorem

How good is the sequence $\{\hat{x}_n\}$ compared to the sequence $\{x_n\}$ when the latter converges linearly? The next theorem shows that it is better, in the sense that it *converges faster*, meaning that the Δ^2-process *accelerates the convergence of* $\{x_n\}$.

[1] Generally, we define $\Delta x_n = x_{n+1} - x_n$ and

$$\Delta^0 x_n = x_n, \quad \Delta^k x_n = \Delta(\Delta^{k-1} x_n), \quad k = 1, 2, \ldots\ .$$

It can be shown by induction on k that

$$\Delta^k x_n = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} x_{n+i}, \quad k = 0, 1, 2, \ldots\ .$$

Theorem 3.1 *Assume that* $\{x_n\}$ *converges to* α *linearly as in* (2.1) *of Definition 2.1. Let* $\hat{x}_n$ *be as in* (2.2)*. Then* $\lim_{n\to\infty} \hat{x}_n = \alpha$ *and*

$$\lim_{n\to\infty} \frac{\hat{x}_n - \alpha}{x_n - \alpha} = 0. \tag{3.1}$$

Proof. Let

$$\epsilon_n = x_n - \alpha, \quad \hat{\epsilon}_n = \hat{x}_n - \alpha, \quad n = 0, 1, \dots\ . \tag{3.2}$$

Subtracting α from both sides of (2.3), we obtain (verify this!),

$$\hat{\epsilon}_n = \frac{\epsilon_n \epsilon_{n+2} - \epsilon_{n+1}^2}{\epsilon_n - 2\epsilon_{n+1} + \epsilon_{n+2}}. \tag{3.3}$$

By (2.1), we can write

$$\lim_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n} = C \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\epsilon_{n+k}}{\epsilon_n} = C^k, \quad k = 1, 2, \dots\ . \tag{3.4}$$

Next, factoring out $\epsilon_n \epsilon_{n+1}$ from the numerator and ϵ_n from the denominator of the right-hand side of (3.3), we obtain

$$\hat{\epsilon}_n = \epsilon_n \frac{\epsilon_{n+1}}{\epsilon_n} \frac{\dfrac{\epsilon_{n+2}}{\epsilon_{n+1}} - \dfrac{\epsilon_{n+1}}{\epsilon_n}}{1 - 2\dfrac{\epsilon_{n+1}}{\epsilon_n} + \dfrac{\epsilon_{n+2}}{\epsilon_n}}. \tag{3.5}$$

Now, by (3.4), and by the fact that $C \neq 1$, which follows from $|C| < 1$,

$$\lim_{n\to\infty} \left(1 - 2\frac{\epsilon_{n+1}}{\epsilon_n} + \frac{\epsilon_{n+2}}{\epsilon_n}\right) = 1 - 2C + C^2 = (1 - C)^2 \neq 0.$$

Dividing both sides of (3.5) by ϵ_n, taking limits, and invoking (3.4) again, we obtain

$$\lim_{n\to\infty} \frac{\hat{\epsilon}_n}{\epsilon_n} = C \frac{C - C}{(1 - C)^2} = 0,$$

and this is simply (3.1). Invoking (3.4) in

$$\hat{\epsilon}_n = \left(\frac{\hat{\epsilon}_n}{\epsilon_n}\right) \epsilon_n,$$

we also have that $\lim_{n\to\infty} \hat{\epsilon}_n = 0$, which implies that $\lim_{n\to\infty} \hat{x}_n = \alpha$. This completes the proof. ∎

4 Application to fixed-point iterative methods for nonlinear equations

The following theorem is a well known result concerning the fixed-point iterative solution of a nonlinear equation of the form $x = \phi(x)$.

Theorem 4.1 *Let α be a solution to the equation*

$$x = \phi(x) \tag{4.1}$$

and assume that $\phi \in C^1(J)$, where $J = [\alpha-\sigma, \alpha+\sigma]$ for some $\sigma > 0$, and assume that $|\phi'(\alpha)| < 1$.

1. *Then, for some arbitrarily chosen m satisfying $|\phi'(\alpha)| < m < 1$, there is a neighborhood $I = [\alpha - \rho, \alpha + \rho]$ of α, $I \subseteq J$, in which (i) $|\phi'(x)| \leq m$, and (ii) α is the unique solution to $x = \phi(x)$.*

2. *The sequence $\{x_n\}$ generated via the fixed-point iterative method*

$$x_{n+1} = \phi(x_n), \quad n = 0, 1, \ldots, \tag{4.2}$$

converges (to α) provided x_0 is in I. Indeed, we have

$$\lim_{n\to\infty} \frac{x_{n+1} - \alpha}{x_n - \alpha} = \phi'(\alpha). \tag{4.3}$$

It is clear that, when $0 < |\phi'(\alpha)| < 1$, $\{x_n\}$ converges linearly; therefore, Theorem 3.1 applies with $C = \phi'(\alpha)$, meaning that the Δ^2-process accelerates the convergence of $\{x_n\}$. In subsection 4.1, we analyze the exact nature of this acceleration assuming that $\phi \in C^2(J)$.

When $\phi'(\alpha) = 0$, provided $\phi \in C^2(J)$ and $\phi''(\alpha) \neq 0$, the sequence $\{x_n\}$ converges quadratically, that is,

$$\lim_{n\to\infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^2} = \frac{1}{2}\phi''(\alpha). \tag{4.4}$$

In this case, the Δ^2-process does *not* accelerate the convergence of $\{x_n\}$, as we show rigorously in subsection 4.2.

4.1 The Δ^2-process and linear convergence

Theorem 4.2 *Let the function $\phi(x)$ be as in the first paragraph of this section with $0 < |\phi'(\alpha)| < 1$ and let $\{x_n\}$ and $\{\hat{x}_n\}$ be as above. Then, provided $\phi \in C^2(J)$, there holds*

$$\lim_{n\to\infty} \frac{\hat{x}_n - \alpha}{(x_n - \alpha)^2} = \frac{1}{2}\frac{\phi'(\alpha)\phi''(\alpha)}{\phi'(\alpha) - 1}, \tag{4.5}$$

and hence

$$\lim_{n\to\infty} \frac{\hat{x}_{n+1} - \alpha}{\hat{x}_n - \alpha} = [\phi'(\alpha)]^2. \tag{4.6}$$

That is, the sequence $\{\hat{x}_n\}$ converges to α linearly, *but* twice as fast as *the sequence $\{x_n\}$.*

Proof. Let us first recall the relation in (3.3), with ϵ_n and $\hat{\epsilon}_n$ as defined in (3.2). Expanding $\phi(x_n)$ in (4.2) about α, we have

$$x_{n+1} = \phi(\alpha)+\phi'(\alpha)(x_n-\alpha)+\frac{1}{2}\phi''(\xi_n)(x_n-\alpha)^2 \quad \text{as } n \to \infty; \quad \xi_n \in \text{int}(x_n, \alpha) \subset I.^2 \tag{4.7}$$

Let us now define

$$\lambda = \phi'(\alpha), \quad \mu = \frac{1}{2}\phi''(\alpha), \quad \mu_n = \frac{1}{2}\phi''(\xi_n). \tag{4.8}$$

Clearly, $\lim_{n\to\infty} \xi_n = \alpha$ and, since $\phi''(x)$ is continuous on I,

$$\mu_n = O(1) \quad \text{as } n \to \infty \quad \text{and} \quad \lim_{n\to\infty} \mu_n = \mu. \tag{4.9}$$

Of course, (4.3) can be expressed as in

$$\lim_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n} = \lambda. \tag{4.10}$$

Recalling also that $\phi(\alpha) = \alpha$, we can re-express (4.7) in the form

$$\epsilon_{n+1} = \lambda\epsilon_n + \mu_n\epsilon_n^2. \tag{4.11}$$

From this, we also have

$$\begin{aligned}\epsilon_{n+2} &= \lambda\epsilon_{n+1} + \mu_{n+1}\epsilon_{n+1}^2 \\ &= \lambda(\lambda\epsilon_n + \mu_n\epsilon_n^2) + \mu_{n+1}(\lambda\epsilon_n + \mu_n\epsilon_n^2)^2 \\ &= \lambda^2\epsilon_n + (\lambda\mu_n + \lambda^2\mu_{n+1})\epsilon_n^2 + O(\epsilon_n^3) \quad \text{as } n \to \infty.\end{aligned} \tag{4.12}$$

[2]Here we have introduced the shorthand notation $\text{int}(a_1, \ldots, a_k) = (\min\{a_1, \ldots, a_k\}, \max\{a_1, \ldots, a_k\})$.

Substituting (4.11) and (4.12) in (3.3), we have, as $n \to \infty$,

$$\begin{aligned}
\hat{\epsilon}_n &= \frac{\epsilon_n[\lambda^2\epsilon_n + (\lambda\mu_n + \lambda^2\mu_{n+1})\epsilon_n^2 + O(\epsilon_n^3)] - [\lambda\epsilon_n + \mu_n\epsilon_n^2]^2}{\epsilon_n - 2[\lambda\epsilon_n + \mu_n\epsilon_n^2] + [\lambda^2\epsilon_n + (\lambda\mu_n + \lambda^2\mu_{n+1})\epsilon_n^2 + O(\epsilon_n^3)]} \\
&= \frac{\epsilon_n[\lambda^2\epsilon_n + (\lambda\mu_n + \lambda^2\mu_{n+1})\epsilon_n^2 + O(\epsilon_n^3)] - [\lambda^2\epsilon_n^2 + 2\lambda\mu_n\epsilon_n^3 + O(\epsilon_n^4)]}{\epsilon_n(\lambda-1)^2 + O(\epsilon_n^2)} \\
&= \frac{\epsilon_n^3[(\lambda^2\mu_{n+1} - \lambda\mu_n) + O(\epsilon_n)]}{\epsilon_n[(\lambda-1)^2 + O(\epsilon_n)]} \\
&= \epsilon_n^2\frac{(\lambda^2\mu_{n+1} - \lambda\mu_n) + O(\epsilon_n)}{(\lambda-1)^2 + O(\epsilon_n)}.
\end{aligned}$$

Dividing both sides by ϵ_n^2, taking limits, and recalling (4.9), we finally obtain

$$\lim_{n\to\infty} \frac{\hat{\epsilon}_n}{\epsilon_n^2} = \frac{\lambda\mu}{\lambda - 1}, \tag{4.13}$$

which proves (4.5). To prove (4.6), we first write

$$\begin{aligned}
\frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_n} &= \frac{\hat{\epsilon}_{n+1}}{\epsilon_{n+1}^2} \cdot \frac{\epsilon_{n+1}^2}{\epsilon_n^2} \cdot \frac{\epsilon_n^2}{\hat{\epsilon}_n} \\
&= \frac{\dfrac{\hat{\epsilon}_{n+1}}{\epsilon_{n+1}^2}}{\dfrac{\hat{\epsilon}_n}{\epsilon_n^2}} \left(\frac{\epsilon_{n+1}}{\epsilon_n}\right)^2,
\end{aligned}$$

which, by taking limits as $n \to \infty$ on both sides, and by invoking (4.13) and (4.10), gives

$$\lim_{n\to\infty} \frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_n} = \lambda^2, \tag{4.14}$$

and this is (4.6). ■

Remark. That $\{\hat{\epsilon}_n\}$ converges twice as fast as $\{\epsilon_n\}$ can be argued heuristically as follows: By the fact that

$$\epsilon_{n+1} \approx \lambda\epsilon_n, \quad \text{when } x_0 \text{ very close to } \alpha,$$

it follows "by induction" that

$$\epsilon_{2n} \approx \epsilon_0\lambda^{2n}.$$

Analogously, by (4.14),

$$\hat{\epsilon}_{n+1} \approx \lambda^2\hat{\epsilon}_n,$$

so that

$$\hat{\epsilon}_n \approx \hat{\epsilon}_0\lambda^{2n}$$

as well. Thus,

$$\hat{\epsilon}_n \approx \left(\frac{\hat{\epsilon}_0}{\epsilon_0}\right)\epsilon_{2n},$$

which shows that $\hat{\epsilon}_n$ and ϵ_{2n} have similar accuracies.

4.2 The Δ^2-process and quadratic convergence

Theorem 4.3 *Let the function $\phi(x)$ be as in the first paragraph of this section with $\phi'(\alpha) = 0$ and let $\{x_n\}$ and $\{\hat{x}_n\}$ be as above. Then, provided $\phi \in C^2(J)$, the sequence $\{\hat{x}_n\}$ converges* quadratically *just as $\{x_n\}$. We have*

$$\lim_{n\to\infty} \frac{\hat{x}_n - \alpha}{(x_n - \alpha)^3} = -\left[\frac{1}{2}\phi''(\alpha)\right]^2. \tag{4.15}$$

and hence

$$\lim_{n\to\infty} \frac{\hat{x}_{n+1} - \alpha}{(\hat{x}_n - \alpha)^2} = -\frac{1}{2}\phi''(\alpha). \tag{4.16}$$

Proof. Using the notation introduced in (3.2) and (4.8), we start by rewriting (4.4) in the form

$$\lim_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n^2} = \mu. \tag{4.17}$$

As a result of (4.17), we also have

$$\lim_{n\to\infty} \frac{\epsilon_{n+1}}{\epsilon_n} = \lim_{n\to\infty} \left[\left(\frac{\epsilon_{n+1}}{\epsilon_n^2}\right)\epsilon_n\right] = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\epsilon_{n+k}}{\epsilon_n} = 0, \quad k = 1, 2, \ldots, \tag{4.18}$$

Then, we rewrite (3.3) in the form

$$\hat{\epsilon}_n = \epsilon_n^3 \frac{\left(\dfrac{\epsilon_{n+1}}{\epsilon_n^2}\right)^2 \left(\epsilon_n \dfrac{\epsilon_{n+2}}{\epsilon_{n+1}^2} - 1\right)}{1 - 2\dfrac{\epsilon_{n+1}}{\epsilon_n} + \dfrac{\epsilon_{n+2}}{\epsilon_n}}. \tag{4.19}$$

Dividing both sides of (4.19) by ϵ_n^3, taking limits as $n \to \infty$, invoking (4.17) and (4.18), we obtain

$$\lim_{n\to\infty} \frac{\hat{\epsilon}_n}{\epsilon_n^3} = -\mu^2, \tag{4.20}$$

which is simply (4.15). To prove (4.16), we proceed as follows: We first write

$$\frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_n^2} = \frac{\dfrac{\hat{\epsilon}_{n+1}}{\epsilon_{n+1}^3}}{\left(\dfrac{\hat{\epsilon}_n}{\epsilon_n^3}\right)^2} \left(\frac{\epsilon_{n+1}}{\epsilon_n^2}\right)^3. \tag{4.21}$$

Taking limits as $n \to \infty$ on both sides of (4.21) and invoking (4.17) and (4.20), we obtain

$$\lim_{n\to\infty} \frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_n^2} = -\mu. \tag{4.22}$$

which is simply (4.6). This completes the proof. ■

Remark. Theorem 4.3 implies that the Δ^2-process does not accelerate quadratic convergence. Comparing the rate of convergence of x_{n+2} used in constructing $\hat{\epsilon}_n$ with that of $\hat{\epsilon}_n$, we realize that the former is even better. This can be shown as follows: By (4.17) and (4.20),

$$\epsilon_{n+2} \sim \mu\epsilon_{n+1}^2 \sim \mu^3\epsilon_n^4 \quad \text{as } n \to \infty, \quad \hat{\epsilon}_n \sim -\mu^2\epsilon_n^3 \quad \text{as } n \to \infty,$$

from which we have

$$\frac{\epsilon_{n+2}}{\hat{\epsilon}_n} \sim -\mu\epsilon_n \quad \text{as } n \to \infty,$$

which also shows that

$$\lim_{n\to\infty} \frac{\epsilon_{n+2}}{\hat{\epsilon}_n} = 0.$$

4.3 The Δ^2-Process and the Steffensen method

In the preceding subsections, we considered the application of the Δ^2-process to a sequence $\{x_n\}$ generated by the fixed-point iterative method $x_{n+1} = \phi(x_n)$, $n = 0, 1, \ldots$, with x_0 given. We emphasize here that the function $\phi(x)$ need not be known (need not even be available as a black box) for this application.

We now consider a different mode of application of the Δ^2-process, in which the function $\phi(x)$ is available. In this application, we start with an initial approximation x_0 to the solution α of the equation $x = \phi(x)$. We then compute x_1 and x_2 via $x_1 = \phi(x_0)$ and $x_2 = \phi(x_1)$. Next, we apply the Δ^2-process to $\{x_0, x_1, x_2\}$ and compute $\hat{x}_0 = (x_0x_2 - x_1^2)/(x_0 - 2x_1 + x_2)$. Finally, we set $x_0 = \hat{x}_0$ and repeat the process indefinitely. That is, we have the following algorithm:

Input $\bar{x}_0$ and set $x_0 = \bar{x}_0$
for $n = 1, 2, \ldots$ **do**
Compute $x_1 = \phi(x_0)$ and $x_2 = \phi(x_1)$
Compute $\bar{x}_n = \dfrac{x_0x_2 - x_1^2}{x_0 - 2x_1 + x_2}$
Set $x_0 = \bar{x}_n$
enddo

It is not difficult to show that the sequence $\{\bar{x}_n\}_{n=0}^{\infty}$ is also the sequence obtained by applying the Steffensen method to the equation $f(x) = 0$ with $f(x) = x - \phi(x)$, namely,

$$\bar{x}_{n+1} = \bar{x}_n - \frac{f(\bar{x}_n)}{\dfrac{f(\bar{x}_n + f(\bar{x}_n)) - f(\bar{x}_n)}{f(\bar{x}_n)}}, \quad n = 0, 1, \ldots,$$

starting with the same $\bar{x}_0$ as above.

5 A further application

The Δ^2-process accelerates also the convergence of sequences $\{x_n\}$, when

$$x_n = \alpha + c\omega^n + d\sigma^n + O(\theta^n) \quad \text{as } n \to \infty, \tag{5.1}$$

where c and d are some nonzero constants and

$$1 > |\omega| > |\sigma| > |\theta|. \tag{5.2}$$

This sequence converges (to α) linearly, since

$$\begin{aligned}
\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{x_n-\alpha} &= \frac{c\omega^{n+1} + d\sigma^{n+1} + O(\theta^n)}{c\omega^n + d\sigma^n + O(\theta^n)} \\
&= \lim_{n\to\infty} \frac{\omega^{n+1}[c + d(\sigma/\omega)^{n+1} + O((\theta/\omega)^n)]}{\omega^n[c + d(\sigma/\omega)^n + O((\theta/\omega)^n)]} \\
&= \omega \lim_{n\to\infty} \frac{c + d(\sigma/\omega)^{n+1} + O((\theta/\omega)^n)}{c + d(\sigma/\omega)^n + O((\theta/\omega)^n)}.
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{x_n-\alpha} = \omega. \tag{5.3}$$

Therefore, the Δ^2-process accelerates the convergence of such sequences, and we have the following refined result:

Theorem 5.1 *Let $\{x_n\}$ be as described in (5.1) and (5.2) of the preceding paragraph, with the notation therein. Then the sequence $\{\hat{x}_n\}$ generated by the Δ^2-process satisfies*

$$\hat{x}_n - \alpha = d\frac{(\omega-\sigma)^2}{(\omega-1)^2}\sigma^n[1 + O(|\sigma/\omega|^n)] \quad \textit{as } n \to \infty. \tag{5.4}$$

Consequently, $\{\hat{x}_n\}$ converges to α, and we have

$$\lim_{n\to\infty} \frac{\hat{x}_{n+1}-\alpha}{\hat{x}_n-\alpha} = \sigma. \tag{5.5}$$

That is, $\{\hat{x}_n\}$ converges linearly, but faster than $\{x_n\}$ since $|\sigma| < |\omega|$.

Proof. To prove (5.4), we use (3.3) with

$$\epsilon_n = c\omega^n + d\sigma^n + O(\theta^n) \quad \text{as } n \to \infty,$$

which is obtained from (5.1). The result follows after some simple algebra. The result in (5.5) follows from (5.4) in the same way (5.3) follows from (5.1) and (5.2). ■

6 Iterated Δ^2-process

In the previous sections, we discussed the application of the Δ^2-process to a general linearly converging sequence $\{x_n\}$ to generate the sequence $\{\hat{x}_n\}$. Naturally, in case the sequence $\{\hat{x}_n\}$ converges linearly, we can apply the Δ^2-process to $\{\hat{x}_n\}$ to obtain $\{\hat{\hat{x}}_n\}$, where

$$\hat{\hat{x}}_n = \frac{\hat{x}_n\hat{x}_{n+2} - \hat{x}_{n+1}^2}{\hat{x}_n - 2\hat{x}_{n+1} + \hat{x}_{n+2}},$$

and the sequence $\{\hat{\hat{x}}_n\}$ converges faster than $\{\hat{x}_n\}$. Of course, we can repeat this process indefinitely as follows:

$$x_n^{(0)} = x_n, \quad n = 0, 1, \dots\ . \tag{6.6}$$

for $p = 0, 1, 2, \dots$ **do**

$$x_n^{(p+1)} = \frac{x_n^{(p)}x_{n+2}^{(p)} - [x_{n+1}^{(p)}]^2}{x_n^{(p)} - 2x_{n+1}^{(p)} + x_{n+2}^{(p)}}, \quad n = 0, 1, \dots\ . \tag{6.7}$$

endfor (p)

Clearly, $x_n^{(1)} = \hat{x}_n$, $x_n^{(2)} = \hat{\hat{x}}_n$, and so on. The $x_n^{(p)}$ can be arranged in a two-dimensional table as follows:

$$\begin{array}{lllll}
x_0^{(0)} & & & & \\
x_1^{(0)} & & & & \\
x_2^{(0)} & x_0^{(1)} & & & \\
x_3^{(0)} & x_1^{(1)} & & & \\
x_4^{(0)} & x_2^{(1)} & x_0^{(2)} & & \\
x_5^{(0)} & x_3^{(1)} & x_1^{(2)} & & \\
x_6^{(0)} & x_4^{(1)} & x_2^{(2)} & x_0^{(3)} & \\
x_7^{(0)} & x_5^{(1)} & x_3^{(2)} & x_1^{(3)} & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Note that, if $x_0, x_1, \dots, x_N$ are the available sequence terms, then the number of columns that can be computed in this table is $\lceil (N+1)/2 \rceil$.

The iterated Δ^2-process can be applied to sequences $\{x_n\}$ obtained from fixed-point iterative solutions of nonlinear equations $x = \phi(x)$, when these sequences converge linearly. The relevant theory behind this application is rather involved. Nevertheless, there is a nice convergence theorem that can be stated in very simple terms as follows:

Theorem 6.2 (Sidi [1, p. 288]) *Let α be the solution of the equation $x = \phi(x)$, and let the function $\phi(x)$ and the sequence $\{x_n\}$ be as in Theorem 4.1* *of Section 4 with the notation therein, subject to* $0 < |\phi'(\alpha)| < 1$ *so that* $\{x_n\}$ *converges linearly. Assume also that* $\phi \in C^\infty(J)$*. Let the* $x_n^{(p)}$ *be generated as in* (6.6) *and* (6.7)*. Then each column sequence* $\{x_n^{(p)}\}_{n=0}^\infty$ *(p fixed) converges linearly, and there holds*

$$x_n^{(p)} - \alpha = O\big([\phi'(\alpha)]^{(p+1)n}\big) \quad \text{as } n \to \infty. \tag{6.8}$$

Clearly, Theorem 6.2 concerns the column sequences $\{x_n^{(p)}\}_{n=0}^\infty$ in the table above. From (6.8) we conclude that each column in the table converges faster than the column preceding it since $|\phi'(\alpha)| < 1$.

References

[1] A. Sidi. *Practical Extrapolation Methods: Theory and Applications*. Number 10 in Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, 2003.