Home Assignment #1

The Heat Equation, Calculus of Variations

Submission date: midnight 28/11/2016

Please submit the assignment to cs236861@gmail.com with subject "HW1 submission". For questions regarding the assignment, please consult with Mr. Gil Shamai (Email: sgils@cs.technion.ac.il).

HW Instructions (in general)

• For implementation exercises, hand in all relevant and documented .m files, and relevant images as .png/.gif files, as well as an external documentation of the results with relevant images. Each coding exercise must include a running script whose name is given in the dry part, and be reasonably internally documented.

• Answers must be printed, not handwritten.

• For curve evolution, show figure examples of the curve evolutions, for several representative cases, at several representative times. Add the matlab code and the figures to the exercises, and explain what is shown.

• Solutions should be electronically submitted (doc / pdf / ps for the dry part). If you do not have a webcourse account, you can send the solution by e-mail.

• Solitary hand in - no couples.

• The maximal score for this HW is 100, including the bonus question.

• In case of doubt - it is always best to ask.

Problem 1

Exercises 6, 10, 11 from Chapter 1 of the book:
(Note: in Ex. 10 you must prove your solution works. You may work in a fashion similar to Ex. 8 in the book, but you cannot assume the solution to Ex. 8 is given)

Exercise 6

• We saw that the length element of a function $y(x)$ is given by $ds = \sqrt{1 + y'^2}dx$. Show that the area element of a function $z(x, y)$ is given by $da = \sqrt{(1 + z_x^2 + z_y^2)}dxdy$.

• Compute the EL equations for the functional $\int da$. (This is Joseph-Louis Lagrange’s result from 1788).

Exercise 10

• What is the kernel for the solution of the general linear heat equation for a signal in an arbitrary dimension $u(x_1, x_2, \ldots, x_N; t)$?

Formally, we say that $u: \mathbb{R}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}$, where $\mathbb{R}^N$ is the $N$ dimensional Euclidean space spanned by the $x_i$ arguments, and $\mathbb{R}_+$ is the ray of positive values that describes time. The arbitrary dimension linear heat equations is $u_t = \Delta u$, where $\Delta u = u_{x_1x_1} + u_{x_2x_2} + \ldots + u_{x_Nx_N}$ is the Laplacian.

Exercise 11

• What is the kernel of the 2D linear affine heat equation

$$u_t = div (M \nabla u)$$

where $M$ is a positive definite, symmetric matrix

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$
• Plot the level sets (also known as iso-contours, or equal-height contours) of the kernel.

**Problem 2**

• Find a the functions that minimize the following functionals
  - The functional \( L = \int_0^1 [3(u')^2 + 2u + 2x] \, dx, u(0) = u(1) = 1. \)
  - The functional \( L = \int_1^2 \frac{(u')^2}{x^3} \, dx, u(1) = 0, u(2) = 1. \)
  - The functional \( L = \int_{\pi/2}^{\pi} \sqrt{2} k \left( \dot{x}^2 + \dot{y}^2 - k^2 (x + y)^2 \right) dt, \)
    \( x(0) = y(0) = 1, x(\frac{\pi}{2\sqrt{2k}}) = y(\frac{\pi}{2\sqrt{2k}}) = 4. \)

Provide the simplest form of the functions that satisfy the EL equations.

**Problem 3**

• Find the Euler Lagrange equation for the functional (assume Dirichlet boundary conditions)

\[
L = \int F(x, u, u_x, u_{xx}) \, dx
\]

Hint: Use integration by parts where needed, in a manner similar to the Euler Lagrange equation with only \( u, u_x \) involved.

**Problem 4 - Bonus (20 points)**

An autonomous car is driving along a single-lane road at constant speed \( v_0. \)
At time \( t = 0 \) a sensor in the car recognizes a red light \( D \) meters ahead.
The autonomous car receives information from the traffic lights database, and knows that the light is going to turn into green in \( T \) seconds (and then remain green).
We decide to approximate the discomfort of the passengers at each moment
as the squared acceleration of the car \((u_{tt})^2\). The car moves in such a way that the total discomfort over time of the passengers is minimized. Given that the car cannot pass in red light, what will be the speed \(v_1\) of the car when it reaches the traffic light? Write the answer in terms of \(v_0\), \(D\), and \(T\).