Home Assignment #2
Differential Geometry
Submission date: midnight 2/12/2015
Submit through the webcourse system, or by mail to cs236861@gmail.com with subject "HW2 submission".

HW Instructions (in general)

- For implementation exercises, hand in all relevant and documented .m files, and relevant images as .png/.gif files, as well as an external documentation of the results with relevant images. Each coding exercise must include a running script whose name is given in the dry part, and be reasonably internally documented.

- Answers must be printed, not handwritten.

- For curve evolution, show figure examples of the curve evolutions, for several representative cases, at several representative times. Add the matlab code and the figures to the exercises, and explain what is shown.

- Solutions should be electronically submitted (doc / pdf / ps for the dry part). If you do not have a webcourse account, you can send the solution by e-mail.

- Solitary hand in - no couples.

- In case of doubt - it is always best to ask.

Problem 1

Let $C(p) = (x(p), y(p), z(p))$ be a smooth regular curve with arbitrary parameterization. Prove the following formula for the curvature $\kappa$ of $C(p)$

$$\kappa(p) = \frac{\|C_p \times C_{pp}\|}{\|C_p\|^3}$$
Problem 2

Let \( C(p) = (x(p), y(p)) \) be a smooth regular curve. Implement the following curve evolutions using an explicit scheme on a non trivial curve of your choice.

1. \( C_t = \kappa \vec{N} \)
2. \( C_t = \vec{N} \)
3. \( C_t = \kappa^{1/3} \vec{N} \)

- For each of the curve evolutions, approximate the curve using a polygon with enough sample points, and draw the evolution curves in several different times.

- For the equi-affine curve evolution (3), empirically show that the flow is indeed invariant to equi-affine transformations of a curve. That is, show that we can apply an equi-affine transformation and the equi-affine curve evolution interchangeably.

- Report on numerical difficulties.

Problem 3

In this question, we will visualize Gaussian and mean curvatures of a surface. Let us be given a surface \( S \) parameterized by

\[ X(u, v) = (x(u, v), y(u, v), z(u, v)), \]

where \((u, v) \in U \subset \mathbb{R}^2\). Assume the parameterization to be regular and smooth.

1. Load from the file \texttt{face.mat} the facial surface. The matrices \( X, Y \) and \( Z \) represent the values of \( x, y \) and \( z \). Assume that the surface is parameterized on the unit square, i.e. \((u, v) \in [0, 1]^2\). The matrix \texttt{MASK} is used for data display (it defines the “region of interest”; surface points
that we do not want to display correspond to \( \text{NaN} \) values in \( \text{MASK} \).

Plot the surface using the following code:

```matlab
h = surf(X,Y,Z.*MASK);
axis image, shading interp, view([0 90]), axis off,
lighting phong, camlight head,
set(h,'FaceColor',[1 1 1] * 0.9, 'EdgeColor', 'none',...
'SpecularColorReflectance',0.1,'SpecularExponent',100);
```

2. Compute two matrices \( K \) representing the Gaussian curvature and \( H \) representing the mean curvature, at every point of the surface (use the results in Tutorial 6). Compute the derivatives numerically. Plot the surface with each point colored according to its curvature, using the following code:

```matlab
h = surf(X,Y,Z.*MASK, K);
axis image, shading interp, view([0 90]), axis off, colormap jet
lighting phong, camlight head
set(h,'SpecularColorReflectance',0.1,'SpecularExponent',100);
```

Produce two separate plots for \( K \) and \( H \).

3. Plot the surface where each point is colored in one of three color according to point classification: planar, elliptic or hyperbolic.

**Problem 4**

Let us be given the curve

\[
C(p) = \begin{cases} 
  (p,0,e^{-1/p^2}) & p > 0 \\
  (p,e^{-1/p^2},0) & p < 0 \\
  (0,0,0) & p = 0
\end{cases}
\]

1. Show that \( C(p) \) is regular for all \( p \), and that \( \kappa(p) \neq 0 \) for \( p \neq 0, p \neq \pm \sqrt{2/3} \) and \( \kappa(0) = 0 \).

2. Show that the limit of the osculating planes as \( p \to 0^+ \) is the plane \( y = 0 \), but the limit of the osculating planes as \( p \to 0^- \) is the plane \( z = 0 \). What can be said about the normal \( N(p) \) at \( p = 0 \)?
3. Show that $\tau$ can be defined so that $\tau \equiv 0$. Is $C(p)$ a planar curve?

**Problem 5**

Consider a smooth surface $M$, and a point $p \in M$. We would like to approximate the surface around $p$ using a second order approximation of the form $z(x, y) = a + bx + cy + dxy + ex^2 + fy^2$. We can choose a coordinate system so that the point $p$ is at the origin, and $a = b = c = 0$.

1. Find the mean and the Gaussian curvatures of the second order surface approximation at the origin (analytically).

2. Prove that it is possible to choose the coordinate system so that $t(u, v) = z(x(u), y(v)) = gu^2 + hv^2$. What are $\kappa_1, \kappa_2$ in terms of $g, h$? What is the geometric meaning of this fact?

**Problem 6**

It is known that the first fundamental form can be used to compute areas and lengths on surfaces.

1. The curve $c(t) = (0, t, \sqrt{1 - t^2})$, $t \in [0, 1]$ is a part of a surface parameterised by $\psi(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$, $(u, v) \in$ unit disc in the plane.

Compute the length of the curve using the first fundamental form.

2. Denote the first fundamental form by:

$$G = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

The normal to a surface at an arbitrary point is defined as usual by $\psi_u \times \psi_v$. Prove that length of the normal is given by $\sqrt{\det(G)}$. 
3. Compute the area of the surface from part 1 using the result from part 2.