236861 Numerical Geometry of Images

Tutorial 4

Differential Geometry I - Curves

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Differential Geometry - Introduction

Differential geometry - study of local properties (and sometimes global..) of curves and surfaces, using methods of differential calculus.

Figure: Examples

- Planar curve
- Curve in $\mathbb{R}^3$
- Surface in $\mathbb{R}^3$
Parametric curves

A parametric curve in $\mathbb{R}^n$ is a map $C: I \mapsto \mathbb{R}^n$ on an interval $I \subseteq \mathbb{R}$ into $\mathbb{R}^n$.

Explicitly, $C$ is written as

$$C(p) = (C_1(p), C_2(p), ..., C_n(p)),$$

where $p \in I$ is the parameter of the curve and $C_1, ..., C_n: I \mapsto \mathbb{R}$.

The curve $C(p)$ is called differentiable if $C_i(p), 1 \leq i \leq n$ are differentiable functions w.r.t. the parameter $p$.

The set of all values $\{C(p); p \in I\}$ is called the trace of the curve.
Examples of parametric curves in $\mathbb{R}^2$ and $\mathbb{R}^3$

Which curve is differentiable and which is not?

(1) $C(p) = (p^3, p^2)$

(2) $C(p) = (\cos(5p), \sin(5p), p)$

(3) $C(p) = (p^3 - 4p, p^2 - 4)$

(4) $C(p) = (p, |p|)$
Tangent and arclength

\textit{Tangent (velocity) vector:} \( C_p(p) = (\dot{C}_1(p), \dot{C}_2(p), \ldots, \dot{C}_n(p)) \)

Points with \( C_p(p) = 0 \) are called \textit{singular}. At regular (non-singular) points of \( C(p) \), the \textit{unit tangent vector} is defined as \( T(p) = \frac{C_p}{\|C_p\|} \).

A differentiable parametric curve \( C(p) \) satisfying \( \|C_p(p)\| \neq 0 \) for all \( p \in I \) is said to be \textit{regular}.

Question: which one of the 4 curves from the previous slide is regular and which is not?

\textit{Arclength:} for a curve parameterized on \((0, P)\), the length from 0 to \( p \) is given by

\[
s(p) = \int_0^p \|C_{\tilde{p}}(\tilde{p})\| \, d\tilde{p} ; \ p \in (0, P)
\]
Arclength parameterization

Let \( r(p) : (0, 1) \mapsto (0, R) \) be a monotonously increasing (decreasing) function with the inverse \( p(r) \).

The curve \( C(r) \equiv C(p(r)) \) has a trace equal to that of \( C(p) \), yet, the corresponding tangent vectors \( C_r(r) \) and \( C_p(p) \) are different\(^1\).

A special case of such a re-parameterization is the arclength \( s(p) \). \( C(s) = C(p(s)) \) is called \textit{arclength parameterization} of the curve. \( C(s) \) also satisfies: \( \| C_s(s) \| = 1 \). It corresponds to traveling along the curve with unit velocity.

\[
s(p) = \int_0^p \| C_{\tilde{p}}(\tilde{p}) \| \, d\tilde{p}, \quad \frac{ds}{dp} = \| C_p \|, \quad C_s = \frac{dC}{ds} = C_p \frac{dp}{ds} = \frac{C_p}{\| C_p \|}
\]

\(^1\) Different parameterizations correspond to traveling along the curve with different speeds.
Re-parameterization example

What are the traces of the following two curves?

\[ D : [0, 2\pi] \rightarrow \mathbb{R}^2 \text{ defined as } D(p) = (r \cos(p), r \sin(p)) \]

\[ C : [0, 2\pi r] \rightarrow \mathbb{R}^2 \text{ defined as } C(s) = \left( r \cos \left( \frac{s}{r} \right), r \sin \left( \frac{s}{r} \right) \right) \]

Their tangent vectors \( D_p \) and \( C_s \):

\[ D_p(p) = (-r \sin(p), r \cos(p)) \Rightarrow \| D_p(p) \| = r \]

\[ C_s(s) = \left( -\sin \left( \frac{s}{r} \right), \cos \left( \frac{s}{r} \right) \right) \Rightarrow \| C_s(s) \| = 1 \]

\[ \Rightarrow C \text{ is the arclength re-parameterization of } D: C(s) = D(p(s)). \]
Normal and Curvature - Planar curves

Let \( C(p) : I \to \mathbb{R}^2 \) be a regular curve, with \( T(p) = \frac{C_p}{\|C_p\|} \).

Its normal is defined by \( N(p) = T(p) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

Let \( C(s) = C(p(s)) \) be the arclength parameterization of the curve. There exist a smooth function \( \alpha : [0, L] \to \mathbb{R} \) such that

\[
T(s) = (\cos \alpha(s), \sin \alpha(s))
\]

\[
C_{ss}(s) = T_s(s) = (-\sin \alpha(s), \cos \alpha(s)) \frac{d\alpha(s)}{ds}
\]

The curvature of \( C(s) \) is defined by \( \kappa(s) = \frac{d\alpha(s)}{ds} \).

Show that: \( C_{ss}(s) = \kappa(s)N(s) \), and \( |\kappa(s)| = \|C_{ss}(s)\| \).
Curvature via radius of osculating circle

In $\mathbb{R}^2$, $|\kappa(s)|^{-1}$ equals the radius of the osculating circle at $C(s)$:
Curvature of planar curves - examples

Straight line  Arclength parameterization of a straight line is given by \( C(s) = as + b, \|a\| = 1 \), therefore its curvature is \( \kappa(s) \equiv 0 \) (prove...)

Circle  Arclength parameterization of a circle of radius \( r \) in \( \mathbb{R}^2 \) is

\[
C(s) = (r \cos \left( \frac{s}{r} \right), r \sin \left( \frac{s}{r} \right))
\]

\[
C_s(s) = \left( -\sin \left( \frac{s}{r} \right), \cos \left( \frac{s}{r} \right) \right)
\]

\[
C_{ss}(s) = \left( -\frac{1}{r} \cos \left( \frac{s}{r} \right), -\frac{1}{r} \sin \left( \frac{s}{r} \right) \right)
\]

Therefore:

\[
|\kappa(s)| = \sqrt{\left( \frac{1}{r} \right)^2 \left( \cos^2(s) + \sin^2(s) \right)} = \frac{1}{r}
\]

What is the sign of \( \kappa(s) \)?
Curvature of an arbitrarily parameterized curve

Let $C(p) : I \rightarrow \mathbb{R}^2$ be a regular curve. $\kappa(p) =$?

$s(p) = \int_0^p \| C_\tilde{p}(\tilde{p}) \| d\tilde{p} \Rightarrow \frac{ds}{dp} = \| C_p(p) \| = \sqrt{x_p^2 + y_p^2}$

$T(p) = (\cos(\alpha(p)), \sin(\alpha(p))) \Rightarrow \alpha(p) = \arctan \frac{y_p(p)}{x_p(p)}$

$\frac{d\alpha(p)}{dp} = \frac{1}{1 + (y_p/x_p)^2} \frac{d(y_p/x_p)}{dp} = \frac{y_{pp}x_p - x_{pp}y_p}{x_p^2 + y_p^2}$

$\kappa(p(s)) = \frac{d\alpha(p(s))}{ds} = \frac{d\alpha(p)}{dp} \frac{dp}{ds} = \frac{d\alpha(p)}{dp} \frac{ds}{dp} = \frac{y_{pp}x_p - x_{pp}y_p}{(x_p^2 + y_p^2)^{3/2}}$

The latter formula can also be obtained by applying the chain rule of differentiation on $C_{ss}(s)$ (see "Numerical Geometry of Images", chapter 2).
Examples

Example 1 Calculate the curvature of a circle $C : [0, 2\pi]$

$$C(p) = (r \cos(p), r \sin(p)) :$$

$$x_p = -r \sin(p), \quad y_p = r \cos(p)$$

$$x_{pp} = -r \cos(p), \quad y_{pp} = -r \sin(p)$$

$$\implies \kappa(p) = \frac{r^2}{r^3} = r^{-1}$$

Example 2 Calculate the curvature of a circle $C : [0, 2\pi]$

$$C(p) = (r \cos(-p), r \sin(-p)).$$

What can you say about the traces of the two curves?
Parameterized curves in $\mathbb{R}^3$

Let $C(s) : [0, L] \to \mathbb{R}^3$ be a regular curve given in arclength parameterization.

The curvature of $C$ at $s$ is defined by $\kappa(s) = \|C_{ss}(s)\|$. Hence, $\kappa(s)$ is always positive.

Its normal unit vector is defined by $N(s) = \begin{pmatrix} C_{ss}(s) \end{pmatrix} / \kappa(s)$.

Note that $N(s)$ is defined only when $\kappa(s) \neq 0$.

Exercise: show that $C_s$ and $C_{ss}$ are orthogonal to each other.

\[ \|C_s\|^2 = 1 \Rightarrow \frac{d}{ds} \|C_s\|^2 = 2 \langle C_s, C_{ss} \rangle = 0 \]

We will say that $s \in [0, L]$ is singular point of order 1 if $\kappa(s) = 0$, and restrict ourselves to curves without such singular points.
Osculating plane and binormal vector

**Osculating plane:** a plane spanned at every point by $C_s$ and $C_{ss}$ (we saw that $C_s \perp C_{ss}$).

**Normal plane:** a plane perpendicular to the tangent vector $T$. The normal vector $N$ belongs to the normal plane.

**Binormal vector** is a unit vector $B$ defined by $B = T \times N$. The vectors $T$, $N$ and $B$ define a local **orthonormal** system of coordinates on the curve.

$T$, $N$ and $B$ are related by the equations:

$$
\begin{pmatrix}
T_s \\
N_s \\
B_s
\end{pmatrix} =
\begin{pmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & -\tau \\
0 & \tau & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B
\end{pmatrix},
$$

known as **Frenet formulas**.
Frenet formulas - proof

\[ T_s = \kappa N \]  
By definition of \( \kappa \).

\[ B_s = \tau N \]  
\( B_s \) is orthogonal to \( B \):

\[ \langle B, B \rangle = 1 \Rightarrow \langle B_s, B \rangle = 0 \]

\( B_s \) is also orthogonal to \( T \): Recall that \( \langle B, T_s \rangle = 0 \)
and so

\[ \langle B, T \rangle = 0 \Rightarrow \langle B_s, T \rangle + \langle B, T_s \rangle = \langle B_s, T \rangle = 0 \]

Therefore, \( B_s \) is parallel to \( N \).

\[ \tau(s) \) defined by \( B_s(s) = \tau(s)N(s) \) is called the torsion of the curve.\(^1\)

\[ N_s = \kappa T + \tau B \]  
Prove! (Differentiate \( N = B \times T \))

\(^1\) Sometimes \( \tau(s) \) is defined by \( B_s(s) = -\tau(s)N(s) \).
Exercise: show that a curve with $\kappa \neq 0$ is planar iff $\tau \equiv 0$.

Let us assume a planar curve in the arclength parameterization, given w.l.o.g. by $C(s) = (x(s), y(s), 0)$. Then,

$$T = (x_s, y_s, 0)/\sqrt{x_s^2 + y_s^2}$$
$$N = (-y_s, x_s, 0)/\sqrt{x_s^2 + y_s^2}$$

and the binormal is constant: $B = (0, 0, 1)$. Therefore, $B_s = 0$ and $\tau = 0$.

Now let $C(s)$ be a curve with $\tau \equiv 0$ and $\kappa \neq 0$. We readily have that $B_s = 0$, and therefore the binormal is a constant vector $B = B_0 = \text{const}$. Therefore,

$$\partial_s \langle C(s), B_0 \rangle = \langle C_s(s), B_0 \rangle = 0.$$

It follows that $\langle C(s), B_0 \rangle = \text{const}$, therefore, $C(s)$ is contained in a plane normal to $B_0$, i.e. $C(s)$ is a planar curve.
Interpretation of curvature and torsion

Both curvature and torsion are invariant intrinsic quantities.

The curvature measures the rate of change in the tangent direction at $s$. By definition, $\kappa(s) > 0$.

The torsion measures the rate of change of the osculating plane of the curve. $\tau(s)$ can be positive or negative.

Physically, a 3D curve can be thought of as being obtained from a straight line by bending and twisting:

- the curvature expresses the bending,
- the torsion expresses the twisting.
Fundamental theorem of the local theory of curves

Given two differentiable functions \( \kappa(s) \neq 0 \) and \( \tau(s) \), for \( s \in I \), there exists exactly one 3D curve (determined up to an Euclidean transformation), for which \( s \) is the arclength, \( \kappa \) is the curvature and \( \tau \) is the torsion.

In other words, \( \kappa \) and \( \tau \) describe completely the local behavior of the curve.
Curvature and torsion

For a curve given in arclength parameterization:

\[ \kappa(s) = \| C_{ss}(s) \|, \quad \tau(s) = - \frac{(C_s \times C_{ss}) \cdot C^{(3)}}{\| C_{ss} \|^2} \]

Proof (using Frenet equations):

\[ C_{ss} = \kappa N \]

\[ C^{(3)} = \kappa_s N + \kappa N_s = -\kappa^2 T + \kappa_s N - \kappa \tau B \]

\[ (C_s \times C_{ss}) \cdot C^{(3)} = -\kappa^2 \tau \]

For arbitrary parameterization:

\[ \kappa(p) = \frac{\| C_p \times C_{pp} \|}{\| C_p \|^3}, \quad \tau(p) = - \frac{(C_p \times C_{pp}) \cdot C^{(3)}}{\| C_p \times C_{pp} \|^2} \]
Exercise

Find curvature and torsion of a helix defined as

\[ C(p) = (a \cos(p), a \sin(p), bp), \quad a > 0, \ b \in \mathbb{R} \]

Solution:

\[ C_p(p) = (-a \sin(p), a \cos(p), b) \]
\[ C_{pp}(p) = (-a \cos(p), -a \sin(p), 0) \]
\[ C^{(3)}(p) = (a \sin(p), a \cos(p), 0) \]

\[ \| C_p(p) \| = \sqrt{a^2 + b^2}, \quad C_p(p) \times C_{pp}(p) = (ab \sin(p), -ab \cos(p), a^2) \]
\[ \| C_p(p) \times C_{pp}(p) \| = a \sqrt{a^2 + b^2} \quad (C_p \times C_{pp}) \cdot C^{(3)} = a^2 b \]

\[ \kappa(p) = \frac{a}{a^2 + b^2}, \quad \tau(p) = \frac{b}{a^2 + b^2} \]

A helix has fixed curvature and torsion.