236861   Numerical Geometry of Images

Tutorial 2

Calculus of variations II, Gauss-Seidel method

Matan Sela

©2015
The Euler-Lagrange equation (reminder)

Given the functional

\[ f(u) = \int_{x_0}^{x_1} F(x, u(x), u'(x)) \, dx \]

with \( F \in C^3 \) and all admissible \( u(x) \) having fixed boundary values \( u(x_0) = u^0 \) and \( u(x_1) = u^1 \).

An extremum of \( f(u) \) satisfies the differential equation

\[ F_u - \frac{d}{dx} F_{u'} = 0 \]

with the boundary conditions \( u(x_0) = u^0 \) and \( u(x_1) = u^1 \).
Special cases of the E-L equation

If the integrand does not depend on the independent variable $x$,

$$f(u) = \int_{x_0}^{x_1} F(u(x), u'(x)) \, dx,$$

for a solution of the E-L equation, the first-order differential equation

$$\frac{d}{dx} \left( F - u' F_{u'} \right) = 0,$$

or

$$F - u' F_{u'} = \text{const},$$

must hold (Beltrami identity).
Proof of the Beltrami Identity

Using the full derivative of $F$ by $x$ we obtain

\[
\frac{dF}{dx} = \frac{\partial F}{\partial u} u' + \frac{\partial F}{\partial u'} u'' + \frac{\partial F}{\partial x} \Rightarrow \\
\frac{\partial F}{\partial u} u' = \frac{dF}{dx} - \frac{\partial F}{\partial u'} u'' - \frac{\partial F}{\partial x}
\]

Multiplying the E-L equation by $u'$ we obtain:

\[
u' \frac{\partial F}{\partial u} - u' \frac{d}{dx} \frac{\partial F}{\partial u'} = 0,
\]

or

\[
u' \frac{\partial F}{\partial u} = u' \frac{d}{dx} \frac{\partial F}{\partial u'}
\]
Plugging the obtained identity into the E-L equation, we get:

\[
\frac{dF}{dx} - \frac{\partial F}{\partial u'} u'' - \frac{\partial F}{\partial x} - \frac{\partial F}{\partial u} u' = 0 \rightarrow \\
\frac{dF}{dx} - \frac{\partial F}{\partial u'} u'' - \frac{\partial F}{\partial x} - u' \frac{d}{dx} \frac{\partial F}{\partial u'} = 0 \rightarrow \\
-\frac{\partial F}{\partial x} + \frac{d}{dx} \left( F - u' \frac{\partial F}{\partial u'} \right) = 0
\]

Using the assumption that \( \frac{\partial F}{\partial x} = 0 \) results in

\[
\frac{d}{dx} \left( F - u' \frac{\partial F}{\partial u'} \right) = 0
\]
Brachistochrone problem

- Formulation and first attempt to prove was done by Galileo, 1638. *Brachistos* - shortest. *Chronos* - time.
- John Bernoulli – 1696 solved the problem and challenged math world (Acta Eroditorium)

> “Given points A and B in a vertical plane, to find the path AB down which a movable point M must, by virtue of its weight, proceed from A to B in the shortest possible time”

- The solution curve is very known by the geometers (cycloid)
- He gave one year for the mathematicians of the time to solve the problem
Solutions were presented by
- Leibniz (received letter 9/6/1696, sent back solution 16/6/1696)
- Newton - sent an anonymous answer the very night..

Figure: Brachistochrone
Let $ds = \sqrt{1 + y'^2} \, dx$ be a length element.

Energy conservation

$$\frac{1}{2} m \left( \frac{ds}{dt} \right)^2 - mgy \equiv E_0 = \frac{m}{2} V_0^2 - mgy_0$$

$$\Rightarrow \left( \frac{ds}{dt} \right)^2 = 2g(\alpha + y), \quad \alpha = \frac{V_0^2}{2g} - y_0,$$

Our integral becomes:

$$T = \int \frac{dt}{ds} ds = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2} \, dx}{\sqrt{2g(\alpha + y)}}$$
Problem 1:

\[ \min \left\{ \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{y + \alpha}} \, dx \right\} \]

\( F_x = 0 \) then the Euler-Lagrange equation is:

\[ H = y' F_{y'} - F = c \]

\[ \frac{y'^2}{\sqrt{1 + y'^2} \sqrt{y + \alpha}} - \frac{\sqrt{1 + y'^2}}{\sqrt{y + \alpha}} = -1 \]

\[ = \frac{1}{\sqrt{1 + y'^2} \sqrt{y + \alpha}} = c \rightarrow \]

\[ \frac{1}{\sqrt{1 + y'^2} \sqrt{y + \alpha}} = \frac{1}{\sqrt{2b}} \quad b > 0 \]
Or, more plainly:

\[(\alpha + y)(1 + y'{}^2) = 2b\]  \hspace{1cm} (1)

Note:

\[F_y = \frac{\sqrt{1+y'{}^2}}{(\alpha + y)^{3/2}} \neq 0,\]  \hspace{1cm} (2)

so we cannot take a shortcut..

Note: \[\implies y' \equiv 0\] are not extremals of our functional because of the physical problem we solve (kinetic energy..).

In order to solve (1), let us denote

\[y'(x) = -\tan \frac{\tau}{2}, \quad -\frac{\pi}{2} < \frac{\tau}{2} < \frac{\pi}{2}.\]

\[\implies \alpha + y = \frac{2b}{1 + y'{}^2}.\]  \hspace{1cm} (3)
\[
\frac{2b}{1 + \tan^2\left(\frac{\tau}{2}\right)} = \frac{2b \cos^2\left(\frac{\tau}{2}\right)}{\cos^2\left(\frac{\tau}{2}\right) + \sin^2\left(\frac{\tau}{2}\right)}
\]

\[= b \left[ 2 \cos^2\left(\frac{\tau}{2}\right) \right] = b(1 + \cos \tau), \tag{4}\]

or

\[y = b(1 + \cos \tau) - \alpha. \tag{5}\]

\[
\frac{dx}{d\tau} = \frac{dx}{dy} \frac{dy}{d\tau} = - \frac{1}{\tan \left(\frac{\tau}{2}\right)}(-b \sin \tau)
\]

\[= b(1 + \cos \tau) \tag{5}\]

Integrating Eq. 5, we obtain

\[\Rightarrow x = a + b(\tau + \sin \tau), -\pi \leq \tau \leq \pi.\]
Problem 2: (Hyperbolic Geodesics)

\[ \min \int_{1}^{2} \frac{\sqrt{1 + y'^2}}{x} \, dx \]

\[ y(1) = 0 \quad y(2) = 1 \]

\[ F = \frac{\sqrt{1 + y'^2}}{x} \]

F is independent of y, and therefore we use

\[ F_{y'} = c \quad \rightarrow \quad \frac{y'}{x\sqrt{1 + y'^2}} = c \]
\[ y' = c^2 x^2 (1 + y'^2) \]
\[ y'^2 (1 - c^2 x^2) = c^2 x^2 \]
\[ y' = \pm \frac{cx}{\sqrt{1 - c^2 x^2}} \]
\[ y = \pm \frac{\sqrt{1 - c^2 x^2}}{c} + c_2 \]
\[ (y - c_2)^2 + x^2 = \frac{1}{c^2} \]

boundary conditions \( \Longrightarrow c = \frac{1}{\sqrt{5}} \quad c_2 = 2 \)

And the solution is:

\[ (y - 2)^2 + x^2 = 5 \]
Problem 3: Constrained maximum

Find a curve $y(x)$ with fixed endpoints $y(\pm 1) = 0$ and perimeter

$$S = \int_{-1}^{-1} \sqrt{1 + y'^2} \, dx,$$

which maximizes the area

$$A(y) = \int_{-1}^{1} y \, dx.$$

Solution

Construct the Lagrangian

$$L(y) = \int_{-1}^{1} y \, dx + \lambda \left( \int_{-1}^{1} \sqrt{1 + y'^2} \, dx - S \right),$$

and denote

$$F(x, y, y') = y + \lambda \sqrt{1 + y'^2}.$$
The Euler-Lagrange equation

\[ 0 = \frac{d}{dx} F_{y'} - F_y = \frac{d}{dx} \left( \frac{\lambda y'}{\sqrt{1 + y'^2}} \right) - 1. \]

Integration w.r.t. \( x \) yields

\[ \frac{\lambda y'}{\sqrt{1 + y'^2}} = x - \alpha, \]

where \( \alpha = \text{const} \). Substitute \( y' = \tan \theta \):

\[ \lambda \frac{y'}{\sqrt{1 + y'^2}} = \lambda \frac{\sin \theta}{\cos \theta} \sqrt{1 + \frac{\sin^2 \theta}{\cos^2 \theta}}, \]

\[ = \lambda \frac{\sin \theta}{\sqrt{\sin^2 \theta + \cos^2 \theta}} = \lambda \frac{\sin \theta}{\cos \theta} \sqrt{\frac{\cos^2 \theta}{1}} = \lambda \sin \theta, \]

from where

\[ \sin \theta = \frac{x - \alpha}{\lambda}. \]
Problem 3: Constrained maximum (cont.)

Substituting again

\[ y' = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\pm \sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{\pm (x - \alpha)}{\lambda \sqrt{1 - \frac{(x \pm \alpha)^2}{\lambda^2}}} = \frac{\pm (x - \alpha)}{\sqrt{\lambda^2 - (x - \alpha)^2}}. \]

Integration w.r.t. \( x \) yields (table of integrals or Mathematica)

\[ y = \sqrt{\lambda^2 - (x - \alpha)^2} + \beta, \]

or

\[ (x - \alpha)^2 + (y - \beta)^2 = \lambda^2 \]

where \( \beta = \text{const} \). The latter equation describes a part of a circle with radius \( \lambda \) centered at \((\alpha, \beta)\). The exact values of the constants are determined using the endpoint conditions and the perimeter constraint.
Problem 4

Note about the Beltrami identity: it merely states a necessary condition for an extremum of the functional.
Q: Show that a solution of the Beltrami identity does not have to solve the Euler Lagrange equation.
A: Look at

$$\int \left( \frac{1}{2} (u')^2 - u \right) \, dx$$

and the function \( u \equiv 1 \), for which the Beltrami identity holds

$$\frac{d}{dx} \left( F - u' F_u' \right) =$$

$$\frac{d}{dx} \left( \frac{1}{2} (u')^2 - u - u'(u') \right) =$$

$$\frac{d}{dx} \left( -\frac{1}{2} (u')^2 - u \right) = -(u')(u'') - u' = 0,$$
and yet,

$$F_u - \frac{d}{dx} F_{u'} =$$

$$-1 - \frac{d}{dx} (u') = -1 - u'' = 0 \rightarrow u'' = -1.$$

In fact, according to the E-L equation, extrema of the functional are of the form

$$u(x) = -x^2 + ax + b$$

This is physically not very surprising – the integrand represents the kinetic energy $T$ of a mass plus its potential energy $-V$, in $1D$. The resulting functions describe a free-fall behavior, subject to initial position and velocity. The functional is known as the action integral, or action.

*Hamilton’s principle* states that the path of a particle (in a system which conserves the total energy, $E = T - V$) must be such that it describes an extremum of the action integral.
A practical example:
Optical flow – Horn and Schunck’s method

Given 2 images, we would like to find a field that translates pixels from one image to the next
In a 1D world, we would look at an image flow field $u(x)$ that moves pixels from $I_1$ to $I_2$. We would like to preserve the brightness between the two images. This is known as the *brightness constancy assumption*. Using 1st order Taylor approximation, we would get:

$$I_1(x + u) \approx I_1(x) + I_{1,x}u.$$  

Using the brightness constancy assumption, we have

$$I_{1,x}u + I_1 - I_2 \approx 0$$

.. but this is only approximate, and our images are usually 2D..
In a 2D image case, we denote the field at each point as \( u(x, y) \), \( v(x, y) \). Taylor approximation of \( I \) now reads:

\[
I_1(x + u, y + v) \approx I_1 + I_{1,x}u + I_{1,y}v.
\]

The brightness constancy assumption gives us the following functional on \( u, v \):

\[
\int_{\Omega} (I_{1,x}u + I_{1,y}v + I_1 - I_2)^2 d\Omega
\]

.. but this is not enough! We need some way to propagate information in the field.. Otherwise, the constraint

\[
l_xu + l_yv = l_t
\]

will not suffice (2 variables, 1 equation).
Since using a single constraint is not enough to determine $u$ and $v$. We therefore add regularization to the functional:

$$
\int_{\Omega} \left[ (I_{1,x}u + I_{1,y}v + I_1 - I_2)^2 + \lambda(|\nabla u|^2 + |\nabla v|^2) \right] d\Omega
$$

Denoting $I_t = I_2 - I_1$, We can write the equation as:

$$
\int_{\Omega} \left[ (I_x^2u^2 + I_y^2v^2 + I_t^2 + 2I_xI_yuv + 2I_xI_tu + 2I_xI_tv) + \lambda(u_x^2 + u_y^2 + v_x^2 + v_y^2) \right] d\Omega
$$
Writing the two Euler-Lagrange equations (for $u$ and $v$), we get:

\[
(2l_x^2 u + 2l_x l_y v) + \lambda \left( \frac{d}{dx} u_x + \frac{d}{dy} u_y \right) = -2l_x l_t
\]

\[
\Delta u
\]

\[
(2l_x l_y u + 2l_y^2 v) + \lambda \left( \frac{d}{dx} v_x + \frac{d}{dy} v_y \right) = -2l_y l_t
\]

\[
\Delta v
\]

We see that this regularization boils down to diffusing the components $u, v$ of the field. This is not a coincidence!
Optical Flow, Horn and Schunck’s method (cont.)

If we add several more tricks, we can obtain a nice result, (after some work...):

Figure: Optical flow (taken from A. Bruhn and J. Weickert, IJCV’05)