236861 Numerical Geometry of Images

Tutorial 5

Differential Geometry II
Surfaces

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Parameterized surfaces

A parameterized surface $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ a differentiable map [1] X from an open set $U \in \mathbb{R}^2$ to $\mathbb{R}^3$.

Explicitly, $X(U)$ is written as

$$X(u,v) = (x(u,v), y(u,v), z(u,v))\,.$$

$X(U) \in \mathbb{R}^3$ is called the *trace* of X.

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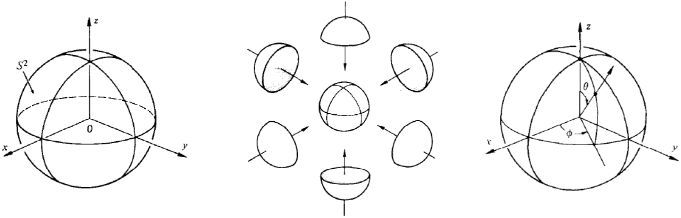
[1] $X(u,v)$ is called *differentiable* if $x(u,v)$, $y(u,v)$ and $z(u,v)$ are differentiable functions w.r.t. the parameters u, v.

Parameterizations of a sphere - examples [2]



(*a*) (*b*) (*c*)

Figure: (a) Sphere in $\mathbb{R}^3$. (b) Six overlapping parameterization of a kind $X(x, y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right)$. (c) $X(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

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[2] Differential geometry of curves and surfaces, by Manfredo P. do Carmo

Differential of X

Let $X : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a differential map. For each $p \in U$ we will define the *differential* of X at p by

$$dX_p : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$$

Let $w \in \mathbb{R}^n$ and let $\alpha : (-\epsilon, \epsilon) \rightarrow U$ be a differential curve such that $\alpha(0) = p$, $\alpha'(0) = w$.

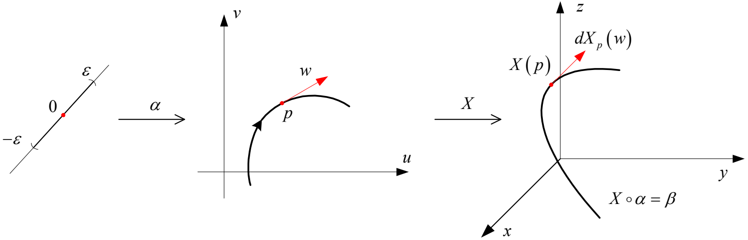
An image of α in $\mathbb{R}^m$ s given by

$$\beta = X \circ \alpha(-\epsilon, \epsilon) \rightarrow \mathbb{R}^m$$

Then:

$$dX_p(w) = \beta'(0)$$

Differential of X: illustration



Calculation of the differential in $\mathbb{R}^3$

Let $X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a differential map.

$$\alpha(t) = (u(t), v(t))$$

$$\beta(t) = X \circ \alpha(t) = \left(x\left(u(t), v(t)\right), y\left(u(t), v(t)\right), z\left(u(t), v(t)\right)\right)$$

$$\beta'(0) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = dX_p(w)$$

Note: $dX_p(w)$ depends only on w, and not on the choice of $\alpha(t)$.

Exercise: dF_p calculation

Calculate dF_p for $F(u,v) = \big(u^2 - v^2, 2uv\big)$, $(u,v) \in \mathbb{R}^2$.

Solution:

$$x = u^2 - v^2, \;\; y = 2uv$$

$$dF_p = \left(\begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right) = \left(\begin{array}{cc} 2x & -2y \\ 2y & 2x \end{array} \right)$$

$$dF_{(1,1)}(2,3) = dF_p \left(\begin{array}{c} 1 \\ 1 \end{array} \right) = (-2, 10)$$

Regular parameterization, Tangent space, Normal

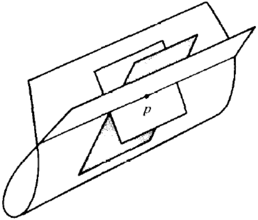
We will say that a differential parameterization $X(U)$ is *regular* if $X_u(u,v)$ and $X_v(u,v)$ are linearly independent for all $(u,v) \in U$.

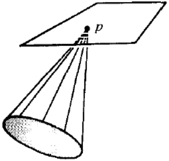
Tangent plane (space) at p: a plane T_p containing all the tangents to curves passing through the point $p \in S$. The tangent plane is spanned by the vectors $X_u(u,v), X_v(u,v)$.

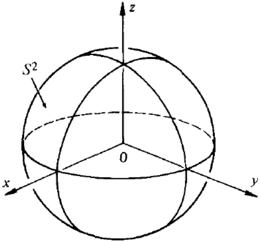
Surface normal: a unit vector orthogonal to the tangent plane at the point p, defined by

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(p)$$

Surfaces in $\mathbb{R}^3$ - examples







Exercise: tangent space, normal

Calculate the equation of the tangent plane of a surface which is a graph of a differentiable function $z = f(x, y)$, at the point (x_0, y_0).

Solution:

$$X(x, y) = (x, y, f(x, y))$$

$$X_x(x, y) = (1, 0, f_x(x, y))$$

$$X_y(x, y) = (0, 1, f_y(x, y))$$

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}$$

Equation of a plane passing through (x_0, y_0) and perpendicular to $N(x_0, y_0)$ is:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

The first fundamental form

In order to measure local distances on the regular surface S, we assign an inner product on the tangent plane T_p of S at every point p.

The *first fundamental form:* a quadratic form $I_p : T_p(S) \times T_p(S) \rightarrow \mathbb{R}$, given by

$$I_p(w_1, w_2) = \langle w_1, w_2 \rangle_p .$$

$<, >_p$ is a standard inner product of $\mathbb{R}^3$; p stands for the tangent plane T_p of the surface S at the point p.

The form $I_p(w_1, w_2)$ is bilinear, symmetric and positive-definite.

The first fundamental form in terms of X_u, X_v

Let $\alpha_1(t) = X\left(u_1(t), v_1(t)\right)$ and $\alpha_2(t) = X\left(u_2(t), v_2(t)\right)$ be two differential curves such that

$$\alpha_1(0) = \alpha_2(0) = p, \;\; \alpha_1'(0) = w_1, \;\; \alpha_2'(0) = w_2.$$

The first fundamental form is given by:

$$I_p(w_1, w_2) = \left\langle \alpha_1'(0), \alpha_2'(0) \right\rangle = \left\langle X_u u_1' + X_v v_1', X_u u_2' + X_v v_2' \right\rangle$$

$$= \begin{pmatrix} u_1' \\ v_1' \end{pmatrix}^T \begin{pmatrix} \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_v, X_u \rangle & \langle X_v, X_v \rangle \end{pmatrix} \begin{pmatrix} u_2' \\ v_2' \end{pmatrix}$$

$$= \begin{pmatrix} u_1' \\ v_1' \end{pmatrix}^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u_2' \\ v_2' \end{pmatrix}$$

The first fundamental form - alternative notation

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_v, X_u \rangle & \langle X_v, X_v \rangle \end{pmatrix}$$

In coordinate notation:

$$I_p(w_1, w_2) = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij} \tilde{w}_1^i \tilde{w}_2^j = g_{ij} \tilde{w}_1^i \tilde{w}_2^j$$

where we use the *Einstein summation convention*, according to which identical sub- and super-scripts are summed up; $\tilde{w}_k = (u'_k, v'_k), k = 1, 2.$

Exercise: calculation of I_p

Calculate the first fundamental form of a unit sphere parameterized by

$$X(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Solution:

$$X_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$X_\phi = (\sin\theta - \sin\phi, \sin\theta\cos\phi, 0)$$

$$E = \langle X_\theta, X_\theta \rangle = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta = 1$$

$$F = \langle X_\theta, X_\phi \rangle = -\cos\theta\cos\phi\sin\theta\sin\phi + \cos\theta\sin\phi\sin\theta\cos\phi = 0$$

$$G = \langle X_\phi, X_\phi \rangle = \sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi = \sin^2\theta$$

Length calculation

The length of a parameterized curve $\beta : [0, T] \to S$ is given by

$$s(t) = \int_0^t \|\beta'(\tilde{t})\| d\tilde{t} = \int_0^t \sqrt{I_p(\beta', \beta')} d\tilde{t}$$

Given that $\alpha(t) = (u(t), v(t))$, $\beta(t) = X \circ \alpha(t)$,

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} d\tilde{t}$$

Also, local element of the arclength is given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

Area calculation

Area of a bounded region R of a regular surface $S = X(U)$ is given by

$$A(R) = \int\int_Q \|X_u \times X_v\| \, dudv, \quad Q = X^{-1}(R)$$

From the equality

$$\|X_u \times X_v\|^2 + \langle X_u, X_v \rangle^2 = \|X_u\|^2 \|X_v\|^2$$

it follows that

$$\|X_u \times X_v\| = \sqrt{EG - F^2}$$

Note that $EG - F^2 > 0$, by definition (prove).

Exercise: arclength and area measured on a sphere

Calculate ds^2 and $dA = \|X_u \times X_v\| \, d\theta d\phi$ of a unit sphere parameterized by

$$X(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Solution: the first fundamental form of a sphere is given by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

Therefore:

$$ds^2 = Ed\theta^2 + 2Fd\theta d\phi + Gd\phi^2 = d\theta^2 + \sin^2\theta d\phi^2$$

$$dA = \sqrt{EG - F^2} d\theta d\phi = \sin^2\theta d\theta d\phi$$

Orientation of surfaces

We saw that a surface normal at point $p \in S$ is defined by

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(p)$$

We will say that a regular surface S is *orientable* if and only if there exist a differentiable field of unit normal vectors $N : S \to \mathbb{R}^3$ on S.

Orientable surfaces: sphere, paper roll.
Non-orientable surface: *Möbius* strip.