Distributed Maximal Matching: Greedy is optimal

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Edge-colored graphs

• There are $k$ colors
  – Each one has a unique number from $\{1, \ldots, k\}$
• The vertices are anonymous
• The coloring is legal
Deterministic distributed algorithms

- We will only consider:
  - Graphs which are already properly colored
- In each *synchronous* round, every node:
  1. Sends a message to each of its neighbors
  2. Receives a message from each of its neighbors
  3. Updates its own state
Deterministic distributed algorithms

• The formal definition:
  A distributed algorithm is a function $A$ that associates a local output $A(V, \nu)$ with any colored graph $V$ and a node $\nu \in V$. 
Algorithms for maximal matchings

• A distributed algorithm $A$ finds a maximal matching in a colored graph $V$ if:
  1. For each $v \in V$ if $A(V, v) = c$ then:
      • There’s an edge $\{u, v\}$ colored with the color $c$, or
      • $c = \perp$
  2. For every two nodes $v, u \in V$ such that the edge $\{v, u\}$ is colored with the color $c$:
      • $A(V, v) = c \iff A(V, u) = c$
      • $A(V, u) = \perp \implies A(V, v) \neq \perp$
      • $A(V, v) = \perp \implies A(V, u) \neq \perp$
Greedy Maximal Matching

- There is a greedy algorithm that solves the problem in $k$ steps:
  - Step $i$:
    - If there’s an adjacent edge colored with color $i$, and
    - the relevant neighbor haven’t chosen any number yet
    - output “$i$”
- Each node runs the same algorithm
Greedy Maximal Matching

• The running time of the greedy algorithm is at most $k - 1$ communication rounds
  – The first step doesn’t require any communication
• The running time of the greedy algorithm is at least $k - 1$ communication rounds
Greedy Maximal Matching

• Correctness:
  – $M$ is a matching
    • The graph is properly colored
  – $M$ is maximal
The Lower bound

• We are going to prove the next theorem:
  – A deterministic distributed algorithm that finds a maximal matching in any anonymous, \( k \)-edged-colored graph requires at least \( k - 1 \) communication rounds.

• The case of \( k = 1 \) is trivial
• In the case of \( k = 2 \), there’s a simple counter example
• When \( k \geq 3 \), it’s more complicated
The lower bound – simple cases

• $k = 1$:
  – There is no need in communication rounds at all.

  ![Diagram for $k = 1$](image)

• $k = 2$:
  – Any algorithm requires at least one round

  ![Diagram for $k = 2$](image)
The Lower bound

• In order to find a counter example for $k \geq 3$, we will use a specific type of graphs
• First of all, we will introduce them
The vertices
The vertices

Given a colored graph $V$ and a vertex $v$, $|v|$ denotes its “distance” from $e$. 
$d$-regular graphs

- A $d$-regular graph is:
  - a colored tree
  - Each node has exactly $d$ neighbors
  - $d \leq k$
$d$-regular graphs

- Given:
  - A $d$-regular graph, $V$
  - A non-negative integer $h$
we define: $V[h] = \{v \in V| h \geq |v|\}$
**d-regular graphs**

- **Given:**
  - A \(d\)-regular graph \(V\)
  - A node \(v \in V\)

We define \(\nu V\) as the same colored tree, with \(\nu\) as \(e\)
**d-regular graphs**

- Let $A$ be a deterministic distributed algorithm
- For every $d$-regular graphs $V, U$ and nodes $v \in V, u \in U$
  - If $vV[r + 1] = uU[r + 1]$, and $A$ stops after at most $r$ communication rounds, then $A(V, v) = A(U, u)$
The lower bound

- **We will now prove that:**
  - Given a set of $k \geq 3$ colors,
  - A deterministic distributed algorithm that finds a maximal matching in any $(k - 1)$-regular graph requires at least $k - 1$ communication rounds.

- **Conclusion:**
  - A deterministic distributed algorithm that finds a maximal matching in any colored graph requires at least $k - 1$ communication rounds.
The lower bound

**Theorem:**

Let $k \geq 3$ be an integer.

Assume that $A$ is a distributed algorithm that finds a maximal matching in any colored graph.

Then there are two $(k-1)$-regular graphs $U$ and $V$ such that:

1. $U[k - 1] = V[k - 1]$
2. $A(U, e) \neq \bot$ and $A(V, e) = \bot$

**Conclusion:**

$A$ requires at least $k - 1$ communication rounds
The lower bound

• Assume that $A$ is a distributed algorithm that finds a maximal matching in any colored graph.

• There exist two graphs $U', V'$ such that:
  
  1. Both $U'$ and $V'$ are $(k - 1)$-regular
  2. $U'[k - 2] = V'[k - 2]
  3. Every node is matched under $A$
  4. There exists a color $y$ such that:
     • $\bar{y}U'[k - 2] = \bar{y}V'[k - 2]
     • $A(U', e) = y \neq A(V', e)$
The lower bound

• Example:
  – $A$ is the greedy algorithm
  – $k = 3$

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2. $U'[k - 2] = V'[k - 2]$
3. Every node is matched under $A$
4. There exists a color $y$ such that:
   \[ yU'[k - 2] = yV'[k - 2] \]
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The lower bound

- Constructing $U, V$:

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The lower bound

• Constructing $U, V$:

$(k - 1)$-regular graphs

$U[k - 1] = V[k - 1]$

$A(U, e) \neq \perp$

$A(V, e) = \perp$
The lower bound

- Constructing $U, V$:

$\begin{align*}
(k - 1)$-regular graphs & \\
U[k - 1] = V[k - 1] & \\
A(U, e) \neq \perp & \\
A(V, e) = \perp
\end{align*}$
The lower bound

**Theorem:**
Let \( k \geq 3 \) be an integer.
Assume that \( A \) is a distributed algorithm that finds a maximal matching in any colored graph.
Then there are two \((k - 1)\)-regular graphs \( U \) and \( V \) such that:

1. \( U[k - 1] = V[k - 1] \)
2. \( A(U, e) \neq \perp \) and \( A(V, e) = \perp \)

**Conclusion:**
\( A \) requires at least \( k - 1 \) communication rounds.
The lower bound

• **Conclusions:**
  • Given a set of $k \geq 3$ colors, a deterministic distributed algorithm that finds a maximal matching in any $(k - 1)$-regular graph requires at least $k - 1$ communication rounds.
  • Holds also for $k \in \{1,2\}$

• **In the general case:**
  • A deterministic distributed algorithm that finds a maximal matching in any colored graph requires at least $k - 1$ communication rounds.