Lifted First-Order Probabilistic Inference

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Presented by Adam Edry in a seminar at the Technion Institute of Technology about Soft Logic in Computer Science, led by Prof. Benny Kimelfeld
8th Jan 2017
Roadmap

• Motivation – Example Problem
• Definitions and Notations
• The Problem Definition
• FOVE – First Order Variable Elimination
• Empirical Results
• Inversion Elimination
• Conclusion
Motivation – Example Problem

Propositional logic

Rules:
• if epidemic then sick(John) 0.7
• If sick(John) then death(John) 0.4
• if epidemic then sick(Mary) 0.7
• If sick(Mary) then death(Mary) 0.4

Queries:
• P(death(John))
• P(death(John)^sick(Mary))
Motivation – Example Problem

Propositional logic

Rules:
• if epidemic then sick(John) 0.7
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• if epidemic then sick(Mary) 0.7
• If sick(Mary) then death(Mary) 0.4

Queries:
• P(death(John))
• P(death(John)^sick(Mary))

What happens when the domain is the entire population of earth?
Motivation – Example Problem

Propositional logic

Rules:
• if epidemic then sick(Person) 0.7
• If sick(Person) then death(Person) 0.4

We use parametric factors - Parfactors

Queries:
• P(sick(Person))
• P(death(John)|sick(John))
• P(death(John)^sick(Mary))
Motivation – Example Problem

Additional Rule:
If death(Person) then someDeath

Query:
P(someDeath)
Motivation – Example Problem

Additional Rule:
If \textit{death(Person)} then \textit{someDeath}

Query:
P(\textit{someDeath} \mid \textit{epidemic})

We want \textit{lifted inference} – inference directly on the first-order level, which instantiates parfactors only when necessary.
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Definitions and Notations

Random Variables in logical expressions:
RV – epidemic and sick.

Truth assignment $\theta \Rightarrow$ expression $epidemic^\wedge sick$ equals:

<table>
<thead>
<tr>
<th>epidemic \ sick</th>
<th>true</th>
<th>false</th>
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<tbody>
<tr>
<td>true</td>
<td>true</td>
<td>false</td>
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<tr>
<td>false</td>
<td>false</td>
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</tbody>
</table>

$sick(John)$ is a ground atom, that is a Boolean random variable
“if epidemic then sick 0.7” is equivalent to:

\[
\phi(\theta) = \begin{cases} 
0.7 & \text{if epidemic} \land \text{sick} \\
0.3 & \text{if epidemic} \land \neg \text{sick} \\
0.5 & \text{otherwise}
\end{cases}
\]

And in general form: “if \( \alpha \) then \( \beta \) \( p \)” is equivalent to:

\[
\phi(\theta) = \begin{cases} 
p & \text{if } \alpha \land \beta \\
1 - p & \text{if } \alpha \land \neg \beta \\
0.5 & \text{otherwise}
\end{cases}
\]
Definitions and Notations

Parfactors: \((\phi, A)\)

- \(A\) – Logical atoms
- \(\phi\) – Potential function on all instantiations of logical atoms in \(A\)

For example, the parfactor:
If \(death(Person1)^{\land}family(Person1, Person2)\) then 
\(grieve(Person2)\) 0.9

Becomes:
\((\phi, \{death(Person1), family(Person1, Person2), griev(Person2)\})\)
Definitions and Notations

• Joint probability distribution over a set of parfactors $G$
  $$P(RV(G)) \propto \prod_{g \in G} \prod_{\theta \in \Theta_g} \phi_g(A_g \theta)$$

• $RV(G)$ - Set of all random variables involved in instantiations of parfactors $g \in G$.

• $\Theta_g$ - All the assignments to the logical variables of $g$.

• $A_g$ - Logical atoms of parfactor $g$.

• $A_g \theta$ - The instantiation of $A_g$ given an assignment $\theta$.

• $\phi_g$ - The potential function in $g$. 
The Problem Definition

Inference:
If $Q$ is a query, the marginal probability of $Q$ given a model $G$ is:

$$P(Q) \propto \sum_{RV(G) \setminus Q} \phi(G)$$

$RV(G) \setminus Q$ are all the assignments to random variables not in $Q$.

$\phi(G)$ is shorthand for $\prod_{g \in G} \prod_{\theta \in \Theta_g} \phi_g(A_g \theta)$.
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FOVE – First Order Variable Elimination

• Variable Elimination (VE) – A method for propositional models.

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<thead>
<tr>
<th>$V_1$</th>
<th>$V_2$</th>
<th>$V_3$</th>
<th>$V_4$</th>
<th>$V_5$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>0.8</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
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<td>T</td>
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$\implies$

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FOVE – First Order Variable Elimination

• Variable Elimination (VE) – A method for propositional models.

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• First Order Variable Elimination (FOVE) – A lifted method that works on first order models. We eliminate one or more atoms in each step.
PROCEDURE \textit{FOVE}(G, Q)

1. If \( RV(G) = Q \), return \( G \).
2. \( G \leftarrow \text{SHATTER}(G, Q) \).
3. \( E \leftarrow \text{FIND} - \text{ELIMINABLE}(G, Q) \).
4. \( G_E \leftarrow \{ g \in G : RV(g) \text{ and } RV(E) \text{ intersect} \} \).
5. \( G_E \leftarrow G \setminus G_E \).
6. \( g' \leftarrow \text{ELIMINATE}(G_E, E) \).
7. \( G' \leftarrow \{ g' \} \cup G_E \).
8. Return \( \text{FOVE}(G', Q) \).
FOVE – First Order Variable Elimination

PROCEDURE $FOVE(G, Q)$
1. If $RV(G) = Q$, return $G$. 
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5. $G_{\overline{E}} \leftarrow G \setminus G_E$. 
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Empirical Results

Query1: \( P(death) \)
From: \{epidemic 0.55, 
    if epidemic then sick(\(X\)) 0.7 else 0.01, 
    if sick(\(X\)) then death 0.55\}
Empirical Results

Query 2: $P(r)$
From: $p(X)^{p(Y)}^r 0.51, X \neq Y$
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Inversion Elimination

We assume $E$ contains a single atom $\{e\}$ such that $LV(e) = LV(G_E)$.

$$\sum_{RV(e)} \phi(G_E)$$
Inversion Elimination

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$$\sum_{RV(e)} \phi(G_E) = \sum_{RV(e)} \phi(g)$$
Inversion Elimination

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$$\sum_{RV(e)} \phi(G_E) = \sum_{RV(e)} \phi(g) = \sum_{RV(e)} \prod_{\theta=\theta_1, \theta_2} \phi_g(A_g \theta)$$
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$$\sum_{RV(e)} \phi(G_E) = \sum_{RV(e)} \phi(g) = \sum_{RV(e)} \prod_{\theta = \theta_1, \theta_2} \phi_g(A_g \theta)$$

$$= \sum_{e \theta_1} \sum_{e \theta_2} \phi_g(A_g \theta_1) \cdot \phi_g(A_g \theta_2)$$
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= \left(\sum_{\theta_1} \phi_g(A_g \theta_1)\right) \cdot \left(\sum_{\theta_2} \phi_g(A_g \theta_2)\right)
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$$= \left( \sum_{e\theta_1} \phi_g(A_g \theta_1) \right) \cdot \left( \sum_{e\theta_2} \phi_g(A_g \theta_2) \right) = \prod_{\theta=\theta_1,\theta_2} \sum_{e\theta} \phi_g(A_g \theta)$$
Inversion Elimination

We assume $E$ contains a single atom $\{e\}$ such that $LV(e) = LV(G_E)$.

\[
\begin{align*}
&= \left( \sum_{e\theta_1} \phi_g(A_g \theta_1) \right) \cdot \left( \sum_{e\theta_2} \phi_g(A_g \theta_2) \right) \\
&= \prod_{\theta = \theta_1, \theta_2} \sum_{e\theta} \phi_g(A' \theta, e\theta)
\end{align*}
\]
Inversion Elimination

We assume \( E \) contains a single atom \( \{e\} \) such that \( LV(e) = LV(G_E) \).

\[
\begin{align*}
\left( \sum_{e\theta_1} \phi_g(A_g \theta_1) \right) \cdot \left( \sum_{e\theta_2} \phi_g(A_g \theta_2) \right) &= \prod_{\theta = \theta_1, \theta_2} \sum_{e\theta} \phi_g(A_g \theta) \\
\prod_{\theta = \theta_1, \theta_2} \sum_{e\theta} \phi_g(A' \theta, e\theta) &= \prod_{\theta = \theta_1, \theta_2} \phi'(A' \theta) = \phi(g')
\end{align*}
\]
We eliminated an atom from the set of parfactors $G_E$ and now express them with one parfactor $g'$.
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Conclusion

FOVE algorithm for lifted first-order probabilistic inference
  • Cheaper inference than the propositional model
  • Inversion elimination and counting elimination

Future research options include:
  • Finding better counting arguments – elimination techniques that are less expensive.
Questions?

Thank you!