Exercise 6.2: Proofs Outline

1. Consider \( n \geq 2 \) players located on the interval \([0, 1]\). Each player knows his location and does not the location of others. A strategy for player \( i \) is a declaration \( s_i : [0, 1] \rightarrow [0, 1] \). A facility location mechanism is a function from declarations to a selected location, i.e. \( m : [0, 1]^n \rightarrow [0, 1] \). Given a selected location \( r \), a player who is in location \( l \) has cost \(|l - r|\). The max-cost of all players given such selected location \( l \) is the maximal cost over all players’ cost (i.e. the worst player’s cost). A mechanism \( m \) is truthful if for any agent \( i \), each location of \( i \), and each declaration of the other agents, declaring the true location by \( i \) minimizes its cost (i.e. truth revealing by all is a dominant strategy equilibrium).

   a. Given \( k \), such that \( 1 \leq k \leq n \), does there exist a mechanism \( m_k \) which is truthful that selects the \( k \)-th left most player’s location? Prove or disprove in detail.

   Yes. Consider a mechanism that selects the \( k \)-leftmost player. Fix the declarations of all players excluding \( i \). Assume that \( i \) cheats about his location. If his real location has been the \( k \)-th leftmost then a deviation will not be beneficial. Otherwise, assume wlog that when \( i \) reports truthfully the \( k \)-th leftmost is to his right, at \( v \). Notice that in order to change this he should report something greater than \( v \) and will move the selected location further away from him (the other case is similar).

   b. We will say that a truthful mechanism \( m \) has \( l \)-approximation to the optimal max-cost, if regardless of the players’ location the max-cost suffered is at most \( l \) times higher than in an optimal location which could be selected (regardless of game-theoretic considerations). What is the best approximation that can be achieved? Prove in detail that your result is tight.

   See the paper ”Approximate Mechanism Design Without Money” in the course site. Theorem 2.2 includes the proof of the non-immediate part.