Exercise 5

1. VCG for $m$ items in general, and its truthfulness.

Let $N = \{1, 2, \ldots, n\}$ be a set of players and $M = \{1, 2, \ldots, m\}$ be a set of items. Any player $i$ has a valuation function $v_i : 2^M \to \mathbb{R}_+$ that satisfies $v_i(\emptyset) = 0$, and $v_i(A) \leq v_i(B)$ for every $A \subseteq B$. Assume that when player $i$ is assigned set of items $T_i$, and needs to pay $p_i$ then his utility is $u_i = v_i(T_i) - p_i$.

The VCG mechanism acts as follows:

1. Consider all allocation functions $A, a : M \to N$ for every $a \in A$. Let $v_i(a) = v_i(\{m \in M : a(m) = i\})$. Players are asked to report their $v_i$’s. Let $b_1, \ldots, b_n$ be the reported valuation functions, respectively. Let $b_i(a) = v_i(\{m \in M : a(m) = i\})$. Let $a_{\text{opt}} \in \arg \max_{a \in A} \sum_{i=1}^n b_i(a)$, and let $V(a_{\text{opt}}) = \sum_{i=1}^n b_i(a_{\text{opt}})$ be the total value of that optimal allocation, and $V^{-i}(a_{\text{opt}}) = \sum_{j \neq i} b_j(a_{\text{opt}})$ be the total value in that allocation of all players excluding $i$. Let $V^{-i} = \max_{a \in A} \sum_{j \neq i} b_j(a)$ be the optimal total value that can be obtained without $i$.

2. Allocate the items according to $a_{\text{opt}}$ (given the declared valuation functions). The payment by player $i$ (again, given the declared valuation functions) is: $V^{-i} - V(a_{\text{opt}}^{-i})$

Write the utility of $i$ when declaring $b_i$ while the others report some $b_{-i}$, and explain why (verbal explanation is enough) reporting $b_i = v_i$ is optimal.

2. Ex-post equilibria for VCG

Given the above setting, let us denote the set of possible valuation functions for player $i$ by $V_i$ (that is, any $v \in V_i$ is $v : 2^M \to \mathbb{R}_+$). A strategy $s_i$ for player $i$ is a mapping that assigns to every valuation $v_i : 2^M \to \mathbb{R}_+$ a declared valuation function $b_i : 2^M \to \mathbb{R}_+$ in $V_i$. Let us denote by $S_i$
the set of possible strategies of $i$. $s = (s_1, \ldots, s_n) \in \Pi_{i=1}^n S_i$ is an **ex-post equilibrium** if for every player $i$, $u_i(s', s_{-i}) \leq u_i(s)$ for every $s' \in S_i$ and for any valuation functions profile of the players. (that is, regardless of what are the real valuation functions of others, the best for each player is to follow the prescribed equilibrium strategy, assuming the others do so).

Notice that being truthful (as above) is an ex-post equilibrium. Prove: there exist ex-post equilibrium of the VCG outcome which is non-truthful (different from $s_i(v_i) = v_i$ for every $v_i \in V_i$ and $i \in N$).

3. Given a sellers-buyers sponsored search setting as discussed is class. Assume 3 advertisers, with values per click $v_1 = 7, v_2 = 6, v_3 = 1$, and 3 slots with click-through rates $c_1 = 10, c_2 = 4, c_3 = 0$.

Find the VCG allocation and payments. Construct an equilibrium of the corresponding GSP auction.

4. Given a sponsored search setting with $n$ advertisers and $n$ slots. Let $v_1 > v_2 > \cdots > v_{n-2} > v_{n-1} > v_n > 0$ be the value per click of advertisers $1, 2, \ldots, n$, respectively, and $c(1) > c(2) > \cdots > c(n-2) > c(n-1) = c(n) = 0$ the click-through rates of slots $1, 2, \ldots, n$, respectively.

A generalized third price auction, called GTP, is defined as follows. Let $b_1 \geq b_2 \geq \cdots \geq b_n$ be the bids of players, then the agent who submitted $b_i$ will receive slot $i$ (breaking tie for the player with lower index), and will pay $b_i + 2c(i)$ where the advertisers who receive slots $n - 1$ and $n$ pay 0.

Prove that GTP has a pure strategy equilibrium.

5. Exercises from book (not for delivery): chapter 15 (1,2,3,4,5)