Introduction to game theory and auctions
(General Overview Slides)

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Outline

Game theory
   Static games
   Dynamic games
   Games with incomplete information

Mechanism design
   From analysis to design
   Auctions
   Some results of auction theory
Game theory

- Game theory deals with the interaction of several self-motivated agents, extending upon classical decision theory.
- In decision theory an agent has to take decision adopting beliefs about the environment:

<table>
<thead>
<tr>
<th>Agent/environment</th>
<th>Rainy</th>
<th>Sunny</th>
</tr>
</thead>
<tbody>
<tr>
<td>Take umbrella</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>Do not take umbrella</td>
<td>-3</td>
<td>3</td>
</tr>
</tbody>
</table>
Game theory

- Game theory deals with the situation where several agents have to take decisions adopting beliefs about one another.

<table>
<thead>
<tr>
<th>Battle of the sexes:</th>
<th>Agent 1/Agent 2</th>
<th>Boxing</th>
<th>Concert</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boxing</td>
<td>(2,1)</td>
<td>(0,0)</td>
<td></td>
</tr>
<tr>
<td>Concert</td>
<td>(0,0)</td>
<td>(1,2)</td>
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Static games

- In a static game each agent has a set of possible strategies to choose from, and a payoff function.
- An agent’s payoff is determined by the strategies selected by all agents.
- Pure coordination game:

<table>
<thead>
<tr>
<th>Agent 1/ Agent 2</th>
<th>Keep the right</th>
<th>Keep the left</th>
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<tbody>
<tr>
<td>Keep the right</td>
<td>(1,1)</td>
<td>(0,0)</td>
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<tr>
<td>Keep the left</td>
<td>(0,0)</td>
<td>(1,1)</td>
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</tbody>
</table>
Static Games in Strategic Form

- A (two-player) game in strategic form is a tuple \(<S_1, S_2, U_1, U_2>\) where \(S_i\) is a set of strategies available to player \(i\), and \(U_i: S_1 \times S_2 \rightarrow \mathbb{R}\) is a utility/payoff function for player \(i\).

- Usually depicted through a payoff matrix
Examples of game in strategic form

- Prisoners’ Dilemma (PD)

<table>
<thead>
<tr>
<th></th>
<th>1,1</th>
<th>3,0</th>
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<tbody>
<tr>
<td>0,3</td>
<td></td>
<td>2,2</td>
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</table>

- The coordination game

<table>
<thead>
<tr>
<th></th>
<th>1,1</th>
<th>0,0</th>
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<tbody>
<tr>
<td>0,0</td>
<td></td>
<td>1,1</td>
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</tbody>
</table>

- Matching pennies

<table>
<thead>
<tr>
<th></th>
<th>1,-1</th>
<th>-1,1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1,1</td>
<td></td>
<td>1,-1</td>
</tr>
</tbody>
</table>
Agent Reasoning in general games

Which route should an agent take?
Taking the route that goes through s is α slower than taking the route the goes through f, but service is split when shared among agents.
Agent Reasoning in general games

- Agents behave according to equilibrium.

<table>
<thead>
<tr>
<th>Agent1/Agent2</th>
<th>f</th>
<th>s</th>
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</thead>
<tbody>
<tr>
<td>f</td>
<td>1/2, 1/2</td>
<td>1, α</td>
</tr>
<tr>
<td>s</td>
<td>α, 1</td>
<td>α/2, α/2</td>
</tr>
</tbody>
</table>
Larger games

- Larger payoff matrix (as many dimensions as players)
- Each dimension has as many entries as there are possible strategies (actions) to that agent; different agents may have different numbers of strategies
“Solving” games

• How will agents behave in a game?
• The solution is not obvious and much of game theory deals with this subject.

• The basic approach: agents will choose an equilibrium strategy. A joint strategy of the agents is in equilibrium if it is irrational for each agent to deviate from it assuming the other agents stick to their part of that joint strategy.
“Solving” games

Defection by all agents is the only equilibrium of the famous prisoners dilemma:

<table>
<thead>
<tr>
<th>Agent 1/ Agent 2</th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>(5,5)</td>
<td>(0,10)</td>
</tr>
<tr>
<td>Defect</td>
<td>(10,0)</td>
<td>(2,2)</td>
</tr>
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A solution concept: the Nash equilibrium.

- A pair of strategies \((s,t)\) is a Nash equilibrium if

\[
\forall (s' \in S_1, t' \in S_2), \quad U_1(s', t) \leq U_1(s, t), \quad U_2(s, t') \leq U_2(s, t)
\]
“Solving” games

Equilibrium analysis enables to analyze games.
In certain cases equilibrium analysis leads to paradoxes, which are then studied and leads to refinements and improvements of solution concepts.
In certain cases an equilibrium is not unique and equilibrium selection becomes a major issue (see the trust game below):

<table>
<thead>
<tr>
<th>Agent 1/ Agent 2</th>
<th>Trust</th>
<th>Do not trust</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trust</td>
<td>(10,10)</td>
<td>(0,8)</td>
</tr>
<tr>
<td>Do not trust</td>
<td>(8,0)</td>
<td>(7,7)</td>
</tr>
</tbody>
</table>
Strategy Types

• Dominant Strategy
  – Best to do no matter what others do
  – e.g., prisoner’s dilemma (PD) has a dominant strategy (best to do no matter what others do)

• The coordination game has several equilibria, but no dominant one.
Mixed Strategies

• Mixed strategies of player $i$: probability distributions on $S_i$, denoted by $\Delta(S_i)$.
  
  – The definition of Nash equilibrium is easily generalized to mixed strategies; rather than look at payoff, look at expected payoff.
“Solving” games

But, does equilibrium always exist?
YES! (Nash), if we allow “mixed” (probabilistic) strategies.

- **Thm.** There always exists a Nash equilibrium in mixed strategies. The result holds also for the case of $n$ players.

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Agent 1 (resp. 2) will give probability $1/3$ to boxing (resp. concert) is an equilibrium of the battle of the sexes.
Decision making in multi-agent systems: the classical game-theoretic approach

- Agents behave according to equilibrium.

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<td>s</td>
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- When α >= 0.5, select f with probability \((2-\alpha)/(1+\alpha)\) is in equilibrium
Zero-Sum Games

- Zero-sum games: $U_1 = -U_2$
  - we will refer only to the payoff of player 1.
- The value of a zero-sum game is the payoff obtained by player 1 in equilibrium, and it is independent of which equilibrium is selected.
Dynamic and multi-stage games

• In a dynamic game there are several stages, and an agent’s strategy may depend on the history of his/her actions taken so far.

• An example: play the battle of the sexes 100 times (e.g. once a day). You may decide to choose boxing on a particular day if and only if concert has been selected by the other agent no more than 5 times.

Extensive form games: agents may alternate in making their moves, e.g. in a revised version of the battle of the sexes agent 1 will make his move to be followed by agent 2’s move.
Repeated Games in Strategic Form

• We play the game finitely or infinitely many times.
• Strategies may depend on the whole history.
• Example 1: finitely repeated PD
  – backward induction.
• Example 2: infinitely repeated PD
  – TFT (tit-for-tat).
Games in Extensive Form: Game Trees

• A two-player game in extensive form is a tree where odd levels are associated with player 1, and even levels are associated with player 2, and the leaves are associated with the players’ payoffs.
• Extensive form games can (in principle) be solved by the minimax algorithm (e.g., Chess).
• In the case of non zero-sum games, this procedure leads to certain paradoxes.
• Example: ...(over)
…the centipede game
Nash- vs subgame-perfect- equilibrium in extensive-form games

• Consider the following game tree

```
    U
   / \  
  D   L
  /    /  
/     /    
R     R
```

• There are two Nash equilibria: (U,R) and (D,L)
• But only one subgame-perfect eqm: (U,R)
Games with incomplete information

There are several ways to model uncertainty in games.

The most classical way to model uncertainty is by referring to partial information about agents’ utility/payoff functions.

The payoff/utility function of an agent is taken to depend on a parameter called the agent’s type, which is typically known to the agent but it is unknown to the other agents.

An agent’s type is for example his willingness to pay for a particular good.

The typical assumption is that the distribution on the agents’ types is known, each agent knows his/her type but in general he/she does not know the type of other agents.
Games with incomplete information

The equilibrium concept has been generalized to games with incomplete information (Harsanyi):

A joint strategy of the agents is in equilibrium, if each agent applies its best response against the strategies of the other agents, given the distribution on agents’ types.
Example: second-price (Vickrey) auction

- In a second-price auction the good will be sold to the agent with the highest bid. He/she will pay the second highest bid.
- In equilibrium of a second-price auction, each agent submits his/her valuation as his/her bid.
- The following strategy is a dominant strategy: truth-revealing is optimal regardless of other agents’ behavior.
Mechanism design: from analysis to synthesis

- Given a description of an environment, i.e. the information structure with regard to the agents’ valuations, the agents’ utility functions (e.g. whether they are risk-neutral, risk-averse, risk-seeking), and an optimization criterion (e.g. maximizing revenue, efficient computation of certain statistics), find an optimal game, a one such that in the equilibrium of which (given the information structure) we will obtain optimal behavior (given the optimization criteria).

- Example: assuming there are n participants, who are risk-neutral with valuations drawn independently and uniformly from the interval \([0,1]\), find an auction procedure that optimizes the seller’s revenue.
Auctions

• Auctions are the most widely-studied economic mechanism.

• Auctions refer to arbitrary resource allocation problems with self-motivated participants.

• The basic insight of auction theory can be generalized and extended to be used in other forms of trade.
Some Classical Assumptions

- Independent valuations for object(s)
- Free disposal
- No Externalities
- Risk-averse/neutral agents (concave utility functions; most analysis is for risk neutral agents who have linear utility functions).
- Constant risk attitude
Auction Rules

- English/Japanese auction
- Dutch auction
- First-price auction
- Second-price auction
- k-price auctions

Combinatorial (multi-dimensional) auctions lead to hard computational problems, but are more expressive
Multi-round auctions lead to complex equilibrium analysis and multiple equilibria
Single-unit English auction

• Bidders call ascending prices

• Auction ends:
  – at a fixed time
  – when no more bids
  – a combination of these

• Highest bidder pays his bid
Multi-unit English auctions

• Different pricing schemes
  – lowest accepted (uniform pricing, sometimes called “Dutch”)
  – highest rejected (uniform pricing, GVA)
  – pay-your-bid (discriminatory pricing)

• Different tie-breaking rules
  – quantity
  – time bid was placed

• Different restrictions on partial quantities
  – allocate smaller quantities at same price-per-unit
  – all-or-nothing
Japanese auction

- Auctioneer calls out ascending prices
- Bidders are initially “in”, and drop out (irrevocably) at certain prices
- Last guy standing gets it at that price
Dutch ("descending clock") auction

- Auctioneer calls out descending prices
- First bidder to jump in gets the good at that price

- With multiple units: bidders shout out a quantity rather than "mine". The clock can continue to drop, or reset to any value.
Sealed bid auctions

• Each bidder submits a sealed bid
• (Usually) highest bid wins
• Price is
  – first price
  – second price
  – k’th price
• Note: Can still reveal interesting information during auction
• In multiple units: similar pricing options as in English
Reverse (procurement) auctions

- English descending
- Dutch ascending
- Japanese descending
Two yardsticks for good auction design

• Revenue: The seller should extract the highest possible price

• Efficiency: The buyer with the highest valuation should get the good
  – usually achieved by ensuring “incentive compatibility”: bidders are induced to bid their true valuation
  – maximizing over those bids ensures efficiency.

• The two are sometimes but not always aligned
Agents care about utility, not valuation

- Auctions are really lotteries, so you must compare expected utility rather than utility.

- Risk attitude speak about the shape of the utility function
  - linear/concave/convex utility function refers to risk-neutrality/risk-aversion/risk-seeking, respectively.

- The types of utility functions, and the associated risk attitudes of agents, are among the most important concepts in Bayesian games (i.e. games with incomplete information), and in particular in auctions. Most theoretical results about auctions are sensitive to the risk attitude of the bidders.
Connections

- Dutch = 1\textsuperscript{st}-price sealed bid

- English ~ Japanese

- English = 2\textsuperscript{nd}-price sealed bid under IPV
Hints about the analysis of auctions

Information assumptions+auctions rules
(+ many other assumptions) yield a
Bayesian game.

• Agents use equilibrium strategies of the Bayesian game.

• We now describe some basic results of the theory of economic mechanism design in order to show the type of studies one can carry. We will emphasize the objective of revenue optimization.
Some Classical Results

- When the agents are risk-neutral, all k-price auctions are revenue equivalent (Myerson).

- When agents are strictly risk-averse, then first-price and Dutch are preferable to second and English (Maskin and Riley, Riley and Samuelson).
Risk-Seeking Agents

- The expected revenue in second-price (English) is greater than the expected revenue in first-price (Dutch).

- The expected revenue in third-price is greater than the expected revenue in second-price (English).

- Under constant risk-attitude:
  
  (k+1)-price is preferable to k-price.
Independent Private Value (IPV) versus Common Value (CV)

- In a CV model agents’ valuations are correlated.
  - the revelation of information during the auction plays a significant role
- In the IPV model they are independent.

- Under CV, risk-neutral bidders, we have that English > 2\text{nd} > 1\text{st}. 
The Revenue Equivalence Theorem

• In all auctions for k units with the following properties
  – Buyers are risk neutral
  – IPV, with values independently and identically distributed over \([a,b]\)
    (technicality – distribution must be atomless)
  – Each bidder demands at most 1 unit
  – Auction allocates the units to the bidders with the k highest valuations
  – The bidder with the lowest valuation has a surplus of 0
• a buyer with a given valuation will make the same expected payment, and therefore
• all such auctions have the same expected revenue
In a revelation mechanism agents are asked to report their types (e.g. valuations for the good), and an action (e.g. decision on the winner and his/her payment) will be based on the agents’ announcement.

In general, agents may cheat about their types, but:

Any mechanism that implements certain behavior (e.g. a good is allocated to the agent with the highest valuation, $v$, and he pays $(1-1/n)v$ (as happens in equilibrium of first-price auction with $n$ risk neutral bidders, whose valuations are uniformly distributed on $[0,1]$) can be replaced by (another) revelation mechanism that implements the same behavior and where truth-revealing is in equilibrium.
The revelation principle: an example

In a first-price auction as before, with only two participants, submitting half of the valuation is in equilibrium.

Consider the following modification: the highest bidder wins, but pays half of his bid.

The new (strange?) protocol implements the same function (the same allocation and payments for every tuple of agents’ valuations), and truth-revealing is in equilibrium there.
Many Participants

- An upper bound -- the expected highest valuation (notice that agents may overbid).

- When the number of participants is large -- English auctions approach the upper bound.

- Marketing is more important than engineering! Attract one more participant and you will increase your revenue more than in selecting an optimized protocol.
Multi-Object Auctions

• Several goods

• Bids may be submitted for subsets of goods

• Valuations need not be additive – the valuation for a set of goods may be different from the sum of valuations for the elements it consists of.

• Agents can do better job in expressing their valuations
Combinatorial bids

- Multiple goods are auctioned simultaneously
- Each bid may claim any combination of goods
- A typical combination: a bundle ("I bid $100 for the TV, VCR and couch")
- More complex combinations are possible
Motivation: complementarity and substitutability

- Complementary goods have a superadditive valuation function:
  - $V({a,b}) > V({a}) + V({b})$
  - In the extreme, $V({a,b}) >> 0$ but $V({a}) = V({b}) = 0$
  - Example: different segments of a flight

- Substitutable goods have a subadditive utility function:
  - $V({a,b}) < V({a}) + V({b})$
  - In the extreme, $V({a,b}) = \text{MAX}[ V({a}) , V({b}) ]$
  - Examples: a United ticket and a Delta ticket
Multi-Object Auctions: 
the Clarke (GVA) Mechanism

• Each agent is asked to reveal its valuation for each subset of the goods
• An optimal allocation (which maximizes the sum of agents’ valuations, given their reported valuations), $O$, is calculated.
• Agent $j$ is required to pay $A_j - B_j$, where $A_j$ is the sum of other agents’ (reported) valuations in an optimal allocation, $O_j$, which ignores $j$, and $B_j$ is the sum of other agents’ (reported) valuations in $O$.
• Truth-revealing is a dominant strategy!
• Notice that in the case of a single good we get the second-price (Vickrey) auction.
Formal definition of GVA

- Each $i$ reports a valuation function $r_i(\cdot)$ possibly different from $v_i(\cdot)$
- The center calculates $(x^*)$ which maximizes sum of $r_i$s
- The center calculates $(\hat{x}_{-i})$ which maximizes sum of $r_i$s without $i$
- Agent $i$ receives $(x_i^*)$ (the goods allocated to it there) and also a payment of

$$\sum_{j \neq i} r_j(x^*) - \sum_{j \neq i} r_j(\hat{x}_{-i})$$
Special case: Multiple units of good (example)

• 2 bidders, 3 units (of a single good)
• Bidder A’s demand curve is (10,8,5), and B’s (9,7,6)
• Outcome:
  – A will win 2 units and B 1 unit
  – A will “get” 9-22=-13, i.e. pay 13 for two goods
  – B will “get” 18-23=-5, i.e. pay 5 for one good
Example (multi-object auction)

• Three goods – A, B, C.
• Three agents 1, 2, and 3.
• The valuation assigned by the agents to the different goods:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>A , B</td>
<td>8</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>A , C</td>
<td>10</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>B , C</td>
<td>10</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>A , B , C</td>
<td>16</td>
<td>14</td>
<td>14</td>
</tr>
</tbody>
</table>

AB goes to 3, and C goes to 1, is optimal, leading to total of 19.
Without 1: BC goes to 2, and A goes to 3, leading to total of 18.
Without 2: as in the optimal allocation.
Without 3: BC goes to 2, and A goes to 1, leading to a total of 17.
Final payments: 1 pays 18-12=6, and 3 pays 17-7=10.
Other remarks about the Clarke (Generalized Vickrey Auction) mechanism

• Applies not only to auctions as we know them, but to general resources allocation problems
  – When “externalities” exist
  – E.g, with public goods
• Not collusion-proof
Multi-Object Auctions: Maximizing Revenue

• An upper bound -- the expected maximal sum of agents’ valuations over all allocations.

• When the number of participants is large, the revenue of the Clarke mechanism approaches the upper bound.
Competition among sellers

- If there are two sellers that use second-price and \( n \) agents, then it is likely that about 50% of the agents will participate at each auction.
- If one of the sellers deviates to a third-price auction, and if an agent prefers second-price to third-price, then more than 50% may participate in the second-price auction.
- However, the expected utility of a buyer is identical in all \( k \)-price auctions, and therefore by deviating to a most profitable auction the seller gains but the buyers do not lose!
- Hence, auctions can serve as legal lotteries!
Motivating Scenario: Mechanism design in networks

Jon wishes to sell his TV to one of $n$ potential buyers.

Highest buyer’s valuation, $v$, is known -- Jon can obtain $v$.

The buyers might not reveal their information in the non-cooperative setup--Jon can use a first-price auction.
Motivating Scenario: mechanism design in networks

Agent 2 might listen to agent 1’s message and submit a slightly higher bid (although his valuation is much higher).

- Agent 1 can send an encrypted bid.
- This might be quite costly.
Motivating Scenario: Game Theory versus Cryptography

A game-theoretic solution:
Use second price instead of first-price --
• The expected revenue in first-price and in second-price auctions are identical.
• In second-price auction it is not beneficial to listen to others’ messages.