Exercise 6 – Proofs Outline

Consider \( n \geq 2 \) players located on the interval \([0, 1]\). The location of each player is his private information, i.e. it is known to him but to no one else. A strategy for player \( i \) is a declaration on his location \( s_i : [0, 1] \rightarrow [0, 1] \). A facility location mechanism is a function from declarations to a selected location, i.e. \( m : [0, 1]^n \rightarrow [0, 1] \). Given a selected location \( r \), a player who is in location \( l \) has cost \(|l - r|\). The social cost of all players given such selected location \( l \) is the sum of their costs. The max-cost of all players given such selected location \( l \) is the maximal cost over all players’ cost (i.e. the worst player’s cost). A mechanism \( m \) is truthful if for any player \( i \), each location of \( i \), and any declaration of the other players, declaring the true location by \( i \) minimizes its cost (i.e. truth revealing by all is a dominant strategy equilibrium).

1. Assume \( n \) is odd. Prove or disprove: for any given \( n \), there exist a mechanism where players are truthful and social cost is minimized for any locations of the players (i.e. compared to the minimal social cost over all possible selections, regardless of that mechanism).

Proof sketch idea: For any given declarations \( p = (p_1 \leq p_2 \leq \cdots \leq p_n) \), wlog (as we can order the players) by players \( 1, 2, \ldots, n \), let the mechanism \( m \) select \( p_q \), the median value of the reported \( p_i \)'s. Naturally a player at location \( p_q \) will not have incentive to deviate given these declarations. As for player \( j \) at location \( l_j \neq p_q \), if declaring \( p_j < l_j \) changes the median value (fixing the other players’ declaration): let \( m_1, m_2 \), where \( m_1 > m_2 \), be the median values for \((p_{-j}, l_j)\) and for \((p_{-j}, p_j)\), respectively, then it must be that \( p_j \leq m_2 < m_1 \leq l_j \), so deviation is not beneficial. Similar analysis holds for \( p_j > l_j \). Hence, we get truthfulness. As for minimizing social cost under the median truthful mechanism: notice that any other location selected which provides
a gain (resp. loss) to player $i, i < \frac{n+1}{2}$ (where players are ordered according to their location as above), comparing to the median location, provides identical loss (resp. gain) to player $i + \frac{n+1}{2}$; hence, one can not obtain lower social cost than the one provided by the median truthful mechanism under any selection of a facility location, even knowing the actual locations.

2. Prove or disprove: for any given $n$, there exist a mechanism where players are truthful and max-cost is minimized for any location of the players (i.e. compared to the minimal max-cost over all possible selections, regardless of that mechanism).

Proof sketch idea: Consider the left most player’s location, $l_a$, and the right most player’s location, $l_b$; then the optimal facility location, minimizing max cost, must be $\frac{l_a + l_b}{2}$, and the max-cost suffered would be $\frac{l_b - l_a}{2}$. This however can not be achieved, as consider the case we have two players, and $l_b = 1$; if $0.5 < l_a = \epsilon > 0$ then the mechanism should select a location that is $\epsilon/2$ to the right of what should be selected if $l_a = 0$; however given any mechanism which is optimal for $l_a = \epsilon$ and $l_b = 1$ then the left-most guy would benefit by declaring 0 as the location selected will be closer to it given the other player report/location is $l_b = 1$.

3. Exercise from book (not for delivery): chapter 23 (exercise 2) and chapter 14(1,2,3,4,5,6).