1. **Greedy Algorithm for Weighted SET-COVER:**

Consider the weighted SET-COVER problem: given a universe \( \{1, \ldots, n\} \) and a family of \( m \) subsets of the universe \( \mathcal{S} = \{S_1, \ldots, S_m\} \) equipped with a non-negative weight function \( w: \mathcal{S} \to \mathbb{R}_+ \), the goal is to choose a collection \( X \subseteq \{1, \ldots, m\} \) of subsets that covers the universe, i.e., \( \bigcup_{j \in X} S_j = \{1, \ldots, n\} \), and minimizes \( \sum_{j \in X} w_j \).

(a) State the greedy algorithm for the weighted SET-COVER problem and prove that it provides an approximation of \( O(\log n) \).

(b) Prove that the greedy algorithm for the weighted SET-COVER problem is tight, i.e., present an instance for which it outputs a cover whose total weight is at least \( \Omega(\log n) \) times the weight of an optimal cover.

2. **The MULTIWAY-CUT Problem:**

Consider the MULTIWAY-CUT problem: given an undirected graph \( G = (V, E) \) equipped with non-negative edge weights \( w: E \to \mathbb{R}_+ \) and a collection \( T = \{t_1, \ldots, t_k\} \subseteq V \) of \( k \) special vertices called terminals, the goal is to find a collection of edges \( F \subseteq E \) that once removed disconnects all terminals, i.e., \( G_F = (V, E \setminus F) \) does not contain any path between \( t_i \) and \( t_j \) for every \( i \neq j \), and minimizes \( \sum_{e \in F} w_e \). Present an algorithm achieving an approximation factor of \( 2(1 - 1/k) \) for the MULTIWAY-CUT problem.

3. **The \( k \)-SUPPLIER Problem:**

Let \( V \) be a collection of \( n \) points equipped with a metric \( d: V \times V \to \mathbb{R}_+ \), i.e., \( d \) satisfies the triangle inequality: \( d(u, v) + d(v, w) \geq d(u, w) \) for every \( u, v, w \in V \). Assume \( V \) is partitioned into suppliers \( S \subseteq V \) and clients \( C = V \setminus S \). Additionally we are given a bound \( k \in \mathbb{N} \). The goal is to choose at most \( k \) suppliers \( X \subseteq S, |X| \leq k \), that minimize: \( \max_{c \in C} \{d(c, X)\} \). Present a 3-approximation for the problem.

4. **Maximization of Monotone Submodular Functions with a Cardinality Constraint:**

Given a universe \( \mathcal{N} \) of size \( n \), a set function \( f: 2^{\mathcal{N}} \to \mathbb{R}_+ \) is submodular if it satisfies:

\[
f(S) + f(T) \geq f(S \cup T) + f(S \cap T) \quad \forall S, T \subseteq \mathcal{N}.
\]

A set function \( f: 2^{\mathcal{N}} \to \mathbb{R}_+ \) is monotone if \( f(S) \leq f(T) \) for every \( S \subseteq T \subseteq \mathcal{N} \).

(a) Prove that an equivalent definition of submodularity is the diminishing returns property:

\[
f(S \cup \{x\}) - f(S) \geq f(T \cup \{x\}) - f(T) \quad \forall S \subseteq T \subseteq \mathcal{N}, \forall x \in \mathcal{N} \setminus T.
\]
(b) Given an undirected graph $G = (V, E)$ equipped with non-negative edge weights $w : E \rightarrow \mathbb{R}_+$, let $\delta : 2^V \rightarrow \mathbb{R}_+$ be the cut function of $G$: $\delta(S) \triangleq \sum_{e=(u,v) \in E; u \in S, v \notin S} w_e$ for every $S \subseteq V$. Prove that $\delta$ is a submodular function but not a monotone one.

(c) Given an auxiliary universe $X$ and a collection of $n$ subsets $\mathcal{S} = \{S_1, \ldots, S_n\}$ of $X$, define $g : 2^\mathcal{S} \rightarrow \mathbb{R}_+$, the coverage function of $X$ by $\mathcal{S}$, as follows: $g(T) \triangleq |\bigcup_{S \in T} S|$. Prove that $g$ is a monotone submodular function.

(d) Given a universe $\mathcal{N}$, a monotone submodular function $f : 2^\mathcal{N} \rightarrow \mathbb{R}_+$, and a cardinality bound $k \in \mathbb{N}$, we want to solve the following problem: $\arg\max_{S \subseteq \mathcal{N} : |S| \leq k} \{f(S)\}$. Prove that the greedy algorithm finds a set $S \subseteq \mathcal{N}$ of size $k$ satisfying: $f(S) \geq (1-1/e) f(S_{\text{OPT}})$ (here $S_{\text{OPT}}$ denotes some optimal solution).

Assume $f$ is not given explicitly but rather the algorithm can access $f$ via a value oracle: given $S \subseteq V$ the oracle return $f(S)$. 